GROUPS OF NEAT AND PURE-HIGH EXTENSIONS OF SOME ABELIAN GROUPS

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IN

MATHEMATICS

By

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I take this opportunity of expressing my deep sense of gratitude to late Professor M.A. Kazim, the former Head, Department of Mathematics, Aligarh Muslim University, who inspite of his indifferent health introduced me to this branch of Algebra and made me acquainted with the new developments in the theory of abelian groups. He was a source of great inspiration to me all the time that I was engaged in this work.

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PREFACE

Homological algebra in recent decades has grown up to be a most important tool in the study of different structures. It was discovered for the first time by Harrison | 6 | that homological algebra plays an important role in the theory of abelian groups. He used homological methods in the discovery of groups of extensions, pure extensions, cotorsion groups and adjusted groups. Harrison | 8 | again exploited homological algebra to investigate the properties of the high, pure-high and neat-high extensions. Later other mathematicians such as Fuchs | 4 | , Irwin |15|, Walker |27|, Yahya |29|, Rangaswamy |22|, Benabdulla | 1 | and many others adopted homological techniques in the development of the abelian group theory.

The study of the group of extensions, pure extensions, neat extensions, pure-high and neat-high extensions of a group A by a group B depends mainly on the groups A and B. Numerous properties of these extensions, in the above mentioned papers were established by using homological methods and by changing the groups A and B. Splitting extensions play an important role in the study of different types of extensions.

The main topics of discussion in this thesis are the neat and pure-high extensions and the groups Next(B,A) and $\text{Next}_p(B,A)$ for some abelian groups A and B. Different types of abelian groups are used and homological methods and techniques are employed for the development of the results. Through out this thesis all groups which have been considered are abelian.

This thesis is divided into five chapters. The principal purpose of the introductory chapter I on preliminaries, is to acquaint the reader with the terminology and the basic results of abelian group theory, which are often used in the subsequent chapters. This chapter is also intended to make the thesis as much self-contained as possible. Here we have given some definitions and properties, specially of divisible groups, pure, neat subgroups and extensions. Exact sequences, commutative diagrams, pure-high and neat-high extensions are also defined.

In chapter II, we have discussed the groups Next(B,A) and $Hext_p$(B,A) by varying the groups A and B from torsion to torsion-free groups. It will be observed that if B is a torsion group then the behaviour of the group $Hext_p$(B,A) is almost the same as that of Next(C,A) for any group C. But if A is a torsion-free group and B is a torsion group, then it will be found that the two groups Next(B,A) and $Hext_p$(B,A) do not behave in a similar way. For instance, Next(Q/Z,A) is a reduced algebraically compact group while $Hext_p$(Q/Z,A) = 0. Then we have studied the groups Next(B,A) and $Hext_p$(B,A) keeping A a torsion-free group and imposing no restriction on the group B. It will be observed that the two groups again do not behave in the same fashion. Finally, we have established that if both the groups A and B are torsion groups, the groups Next(B,A) and $Hext_p$(B,A) have similar properties.

In chapter III, the group Next(B,A) and $Hext_p$(B,A) has been studied by varying the groups A and B from cyclic groups of prime order to direct sum of cyclic groups of prime and prime power order. Exact sequences connecting Next to Hom (theorem 3.2) and relating Hom to Ext to Next (theorem 3.4) have been established. Two subgroups of Next(B,A), in case A is a direct sum of cyclic groups of prime power order have been constructed and the properties of these subgroups and their quotient groups have been studied.

In chapter IV, we have concentrated on finding out a group G that is isomorphic to the group of pure-high extensions for suitable groups A and B. Such groups have been named as Next_p-groups. It has been shown that an elementary p-group is a Next_p-group. Also, all these groups whose pure-high extensions by torsion groups are splitting (that is H_p^t - groups) have been studied. Some of the important properties of H_p^t - groups have also been recorded in this chapter.

Chapter V is devoted to studying the pure-high and neat-high extensions. Conditions under which a K-neat- high extension reduces to a K-pure-high extension and a neat extension coincides with a pure extension have been discussed. Splitting conditions for neat and pure-high extensions, condition on A and B under wich Next(B,A) = 0 and $\mathrm{Next}_p(B, A) = 0$ have been deduced and discussed.

CHAPTER-I

PRELIMINARIES

The main purpose of this introductory chapter is to recall some of the definitions, results and other necessary bits of information which will be used frequently in the subsequent chapters. This is being done only to fix up the terminology and notations needed in the sequel and to make this work selfcontained as far as possible. The contents of this chapter are not original and most of the definitions and results are from | 4 |, | 6 | and | 8 |.

In section 1, some basic concepts definitions, elementary properties and results, mainly of divisible subgroups, pure subgroups and neat subgroups are given.

In section 2, various types of exact sequences, commutative diagrams, necessary bits of information and results for the groups of extensions and pure extensions, through homological methods are given. The important concepts of algebraically compact and cotorsion groups with their elementary properties may also be found in this section.

In section 3, high, pure-high, neat and neat-high extensions are defined. Some of the known basic results which will be used in the subsequent chapters are stated without proof.

1- Some Elementary Concepts.

Definition(1.1). Groups in which every element has a finite order are called torsion groups, those in which all the elements $\neq 0$ are of infinite orders are called torsion-free groups.

Definition(1.2). A group in which the orders of the elements are powers of the same prime number p is called a primary or p-group.

Proposition(1.3). Every torsion group is a direct sum of p-groups for various $p \in P$.

Proposition(1.4). If B_t is the maximal torsion subgroup (torsion part) of B, then B/B_t is torsion-free.

Definition(1.5). Groups in which every element has square-free order are called elementary.

Proposition(1.6). Every elementary p-group is a direct sum of cyclic groups of the same order p.

Proposition(1.7). A group A is elementary if and only if every subgroup of A is a direct summand.

Definition(1.8). The Frattini subgroup $\phi(G)$ of G is the intersection of all the maximal subgroups of G, that is,

$$\phi(G) = \bigcap_{p \in P} pG$$

Definition(1.9). The first Ulm subgroup G' of G is the intersection of all the subgroups of G, that is ,

$$G' = \bigcap_{n \in N} nG.$$

Definition(1.10). The quotient group G/G' is called the 0th Ulm factor of G and is denoted by G_0.

Definition(1.11). A subgroup E of a group A is called essential if $E \cap B \neq 0$, whenever B is a non-zero subgroup of A.

Proposition(1.12). A is elementary if and only if A is the only essential subgroup in A.

Definition(1.13). By a free abelian group is meant a direct sum of infinite cyclic groups. If these cyclic groups are generated by elements $x_i (i \in I)$, then the free group F will be,

$$F = \bigoplus_{i \in I} \langle x_i \rangle \quad .$$

Definition(1.14). A group G is called divisible if $nG = G$ for all $n \in N$.

It follows from the definition that an epimorphic image of a divisible group is divisible and a direct sum or a direct product of groups is divisible if and only if each component is divisible. Furthermore, if $D_i (i \in I)$ are divisible subgroups of A, then so is their sum ΣD_i.

<u>Proposition</u>(1.15). Every group G has a unique decomposition $G = dG \oplus R$, where dG is the maximal divisible subgroup of G. If $dG = 0$, then the group G is called reduced.

<u>Proposition</u>(1.16). A divisible subgroup D of a group A is a direct summand of A, that is $A = D \oplus C$ for some subgroup C of A.

<u>Proposition</u>(1.17). Every group can be embedded as a subgroup in a divisible group.

<u>Definition</u>(1.18). A minimal divisible group E containing a group A is called the divisible hull(injective hull) of A.

<u>Definition</u>(1.19). A subgroup H of G is called a pure subgroup if the equation $nx = h$, where $h \in H, n \in N$ is solvable in H whenever it is solvable in G, or equivalently, if ,

$$nH = H \cap nG$$

holds for all natural numbers n.

It is easy to verify that every direct summand is a pure subgroup. O and A are pure subgroups of A. The torsion part of a mixed group and its p-components are pure subgroups. These fail in general, to be direct summands. It is to be noted that Q and $Z(p^\infty)$ have no pure subgroups. In torsion-free groups, intersection of pure subgroups is again pure. Purity is an inductive property.

Theorem(1.20). Let B, C be subgroups of A such that $C \leq B \leq A$, then we have,

(1) if C is pure in B, and B is pure in A, then C is pure in A,

(2) if B is pure in A, then B/C is pure in A/C,

(3) if C is pure in A, B/C is pure in A/C, then B is pure in A.

Theorem(1.21). If B is a pure subgroup of A such that A/B is a direct sum of cyclic groups, then B is a direct summand of A.

Proposition(1.22). If H is a subgroup of G such that the factor group G/H is torsion-free, then H is pure in G.

Theorem(1.23). For a subgroup B of a group A the following conditions are equivalent,

(1) B is pure in A,

(2) B/nB is a direct summand of A/nB, for every $n > 0$,

(3) if $C \leq B$ is such that B/C is finitely cogenerated, then B/C is a direct summand of A/C.

Theorem(1.24). The following are equivalent conditions for a subgroup B of A,

(1) B is pure in A,

(2) F is a direct summand of $n^{-1}B$, for every $n > 0$,

(3) If C is a group between B and A such that C/B is finitely generated, then F is a direct summand of C.

It is to be noted that a group is pure in every group containing it exactly if it is divisible.

Theorem(1.25). If G is a pure subgroup of the group A, then,

(1) $G' = G \cap A'$,

(2) $(G + A')/A'$ is pure in A/A',

(3) $G \leq A'$ implies G is divisible.

Definition(1.26). A subgroup H of a group G is called a neat subgroup if the equation $px = h \in H, p \in P$, is solvable in H whenever it is solvable in G. This is equivalent to the requirement that,

$$pH = H \cap pG ,$$

for all $p \in P$.

Every pure subgroup of a group G is neat, but every neat subgroup of G need not be pure. Neatness is a transitive property, it is an inductive property, that is, the union of an ascending chain of neat subgroups is itself a neat subgroup. In torsion-free groups neatness is equivalent to purity.

Example(1.27). The intersection of two or more neat subgroups of a group is not, in general, neat. Let $G = \{a\} + \{b\}$ where a is of order p^3 while b is of order p. Then the subgroups $A = \{a\}$ and $B = \{(pa + b)\}$ are neat in G. But the subgroup $C = A \cap B = \{p^2a\}$ is not neat in G. It is to be noted that B is not pure in G, since it contains p^2a of height 2.

Theorem(1.28). If A is a subgroup of G, the following statements are equivalent,

(1) A is a neat subgroup of G,

(2) A is maximal disjoint from some subgroup K of G,

(3) if K is a subgroup of G maximal disjoint from A, then A is maximal disjoint from K,

(4) $A \cap pG = pA$, for all $p \in P$.

Proposition(1.29). If N is a neat subgroup of G and either N itself is an elementary p-group or the factor group G/N is elementary, then N is a direct summand of G.

Proposition(1.30). If E is a minimal divisible group containing G, then N is a neat subgroup of G if and only if $N = G \cap D$ where D is a divisible subgroup of E.

Proposition(1.31). If K and L are groups then Hom(K,L) denotes the group of homomorphisms of K into L.

If K is an infinite cyclic group, then $\mathrm{Hom}(K,L) \cong L$, and if K is a cyclic group of finite order n then,

$$\mathrm{Hom}(K,L) \cong L[n].$$

Proposition(1.33). In each of the following cases $\mathrm{Hom}(K,L) = 0$,

(1) if K is a torsion group and L is a torsion-free group,

(2) if K is a divisible group and L is a reduced group,

(3) if K is a p-group and L is a q-group, $p \neq q$.

Theorem(1.34). If K and L are groups the following hold,

(1) $\mathrm{Hom}(K,L)$ is a torsion-free group, whenever L is a torsion-free group,

(2) if L is torsion-free and divisible, then $\mathrm{Hom}(K,L)$ is torsion-free and divisible,

(3) if K is divisible, then $\mathrm{Hom}(K,L)$ is torsion-free,

(4) if K is torsion-free and divisible then so is $\mathrm{Hom}(K,L)$.

(5) if K is torsion-free and L is divisible, then $\mathrm{Hom}(K,L)$ is divisible.

Theorem(1.35). There exist natural isomorphisms,

$$\mathrm{Hom}(\bigoplus_{i\in I} K_i, G) \cong \prod_{i\in I} \mathrm{Hom}(K_i, G),$$

$$\mathrm{Hom}(K, \prod_{i\in I} L_i) \cong \prod_{i\in I} \mathrm{Hom}(K, L_i).$$

2-Exact Sequences and Extensions.

Definition(1.36). The sequence ,

$$\cdots\cdots \longrightarrow A_{k-1} \xrightarrow{\varphi_{k-1}} A_k \xrightarrow{\varphi_k} A_{k+1} \longrightarrow \cdots\cdots$$

of groups and homomorphisms is called an exact sequence if the image of each homomorphism is equal to the kernel of the next homomorphism.

If the sequence ,

$$0 \longrightarrow A \xrightarrow{\alpha} B$$

is exact then α is a monomorphism.

If the sequence ,

$$B \xrightarrow{\beta} C \longrightarrow 0$$

is exact then β is an epimorphism.

The exactness of the sequence ,

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is equivalent to the fact that α is an isomorphism.

The exact sequence of the form ,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a short exact sequence.

The short exact sequence,

$$0 \longrightarrow A \longrightarrow D \longrightarrow D' \longrightarrow 0$$

with D(and hence D') divisible, which exists for every group A, is called an injective resolution of A.

The short exact sequence,

$$0 \longrightarrow H \longrightarrow F \longrightarrow A \longrightarrow 0$$

with F(and hence H) free,which exists for every group A, is called the free or projective resolution of A.

<u>Definition</u>(1.37). If A,B,C,and D are groups and α, β, ν, δ, μ are homomorphisms, then the diagrams,

$$\begin{tikzcd} & C \arrow[d, "\alpha"] \\ A \arrow[ur, "\nu"] \arrow[r, "\mu"'] \arrow[dr, "\delta"'] & B \arrow[d, "\beta"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\nu"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\delta"'] & D \end{tikzcd}$$

are commutative if,

$$\alpha\nu = \mu \quad , \beta\mu = \delta \text{ and } \beta\alpha = \delta\nu$$

respectively.

<u>The five lemma</u>(1.38).Consider the commutative diagram with exact rows of groups,

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\alpha_1"'] & A_2 \arrow[r] \arrow[d, "\alpha_2"'] & A_3 \arrow[r] \arrow[d, "\alpha_3"'] & A_4 \arrow[r] \arrow[d, "\alpha_4"'] & A_5 \arrow[d, "\alpha_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

(1) if α_2 and α_4 are onto and α_5 is one-to-one, then α_3 is onto,

(2) if α_2 and α_4 are one-to-one and α_1 is onto, then α_3 is one-to-one.

<u>The 3×3 lemma</u>(1.39). Assume that the diagram;

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\lambda_1"'] & B_1 \arrow[r, "\beta_1"] \arrow[d, "\mu_1"'] & C_1 \arrow[r] \arrow[d, "\nu_1"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\lambda_2"'] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\mu_2"'] & C_2 \arrow[r] \arrow[d, "\nu_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "\alpha_3"] \arrow[d] & B_3 \arrow[r, "\beta_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

is commutative and all the three columns are exact.
If the first two or the last two rows are exact, then the remaining row is also exact.

It is to be noted that the lemma can be proved without λ_1 and μ_1 being monomorphisms.

<u>Definition</u>(1.40). An extension $G=(G,f)$ of a group A by a

group B is a pair consisting of a group G and a homomorphism f such that ;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is an exact sequence, where i may stand for the identity mapping.

For any two groups A and B, Ext(B,A) can be defined as equivalence classes of short exact sequence;

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

with addition being Baer addition (see|2|).

<u>Theorem</u>(1.41). If,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence and G is any group, then the following sequences are exact;

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)$$

$$\longrightarrow \mathrm{Ext}(G,A) \longrightarrow \mathrm{Ext}(G,B) \longrightarrow \mathrm{Ext}(G,C) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G)$$

$$\longrightarrow \mathrm{Ext}(C,G) \longrightarrow \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G) \longrightarrow 0 .$$

__Theorem__(1.42). Let $\{G_i ; i\in I\}$ be any family of groups, then the following isomorphisms hold;

$$Ext(B, \prod_{i\in I} G_i) \cong \prod_{i\in I} Ext(B, G_i) ,$$

$$Ext(\bigoplus_{i\in I} G_i, A) \cong \prod_{i\in I} Ext(G_i, A).$$

__Definition__(1.43). The exact sequence;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is called splitting exact if iA is a direct summend of G, and (G,f) is called a splitting extension of A by B.

Every extension of A by B splits if and only if $Ext(B,A) = 0$.

__Proposition__(1.44). In each of the following cases $Ext(B,A) = 0$;

(1) if A is a divisible group ;

(2) if B is a free group ;

(3) if A is p-divisible and B is a p-group.

__Definition__(1.45). The exact sequence;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is called pure exact if iA is a pure subgroup of G and (G,f)

is called a pure extension of A by B.

The pure extensions of A by B form a subgroup Pext(B,A) of Ext(B,A) which coincides with the Ist Ulm subgroup of Ext(B,A), that is :

$$\mathrm{Pext}(B,A) = \bigcap_{n \in N} n\,\mathrm{Ext}(B,A) \;.$$

<u>Theorem</u>(1.46). If;

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a pure exact sequence, then for any group G the following sequences are exact;

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)$$

$$\longrightarrow \mathrm{Pext}(G,A) \longrightarrow \mathrm{Pext}(G,B) \longrightarrow \mathrm{Pext}(G,C) \longrightarrow 0 \;,$$

and

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G)$$

$$\longrightarrow \mathrm{Pext}(C,G) \longrightarrow \mathrm{Pext}(B,G) \longrightarrow \mathrm{Pext}(A,G) \longrightarrow 0.$$

<u>Theorem</u>(1.47). Let $\{G_i \,;\, i \in I\}$ be a family of groups, then the following isomorphisms hold;

$$\mathrm{Pext}(B, \pi_{i \in I} G_i) \cong \pi_{i \in I} \mathrm{Pext}(B,G_i),$$

$$\mathrm{Pext}(\bigoplus_{i\in I} G_i, A) \cong \prod_{i\in I} \mathrm{Pext}(G_i, A).$$

<u>Proposition</u>(1.48). In each of the following cases $\mathrm{Pext}(B,A) = 0$:

(1) *for all A, if and only if B is a direct sum of cyclic groups;*

(2) for all B, if and only if $A \cong D \oplus S$, where D is a divisible group and S is a direct summand of a direct product of finite cyclic groups ;

(3) if and only if A is algebraically compact for all B.

<u>Definition</u>(1.49). A group G is called algebraically compact if it is a direct summand of every group which contains it as a pure subgroup. This is equivalent to the requirement that G is algebraically compact if and only if $\mathrm{Pext}(B,G) = 0$ for all groups B.

Using homological methods it is not difficult to prove that the conditions $\mathrm{Pext}(Q/Z,G) = 0$ and $\mathrm{Ext}(Q,G) = 0$ imply that $\mathrm{Pext}(B,G) = 0$ for all groups B. Thus a group G is algebraically compact if and only if ;

$$\mathrm{Pext}(Q/Z,G) = 0 \text{ and } \mathrm{Ext}(Q,G) = 0.$$

Divisible groups are algebraically compact,a group is algebraically compact if and only if its reduced part is algebraically compact.

__Proposition__(1.50).Every group can be embedded as a pure subgroup in an algebraically compact group.

__Proposition__(1.51).If A is algebraically compact, then its first Ulm subgroup coincides with its maximal divisible subgroup.

__Proposition__(1.52).If A is torsion-free,Ext(C,A) is algebraically compact, whenever C is.

__Proposition__(1.53).If A is algebraically compact, then Ext(C,A) is a reduced algebraically compact group.

__Proposition__(1.54).If A is a torsion group, then Hom(A,C) is a reduced algebraically compact group,for any C.

__Proposition__(1.55).A group G is called cotorsion if all of its extension by torsion-free groups are splitting,that is if Ext(B,G) = O for all torsion-free groups B. So a group G is cotorsion if Ext(Q,G) = O.

Every algebraically compact group is cotorsion , conversely, a torsion-free cotorsion group is algebraically compact.

Theorem(1.56). Every cotorsion group G has a unique decomposition into a direct sum of three groups,

$$G = D \oplus A \oplus B ,$$

where D is a divisible group, A a reduced torsion-free cotorsion group, and B is a reduced cotorsion group having no torsion-free direct summand $\neq 0$.

Proposition(1.57). A group is cotorsion if and only if it is an epimorphic image of an algebraically compact group.

Proposition(1.58). A reduced cotorsion group is algebraically compact if and only if its first Ulm subgroup vanishes.

Proposition(1.59). $\mathrm{Ext}(C,A)$ is cotorsion for all groups A and C.

Propostion(1.60). If G is a cotorsion group, then $\mathrm{Hom}(A,G)$ is cotorsion for any A.

3- High Extensions.

Definition(1.61). Let G be a group and H is a subgroup of G maximal with respect to $H \cap G' = 0$, then H is called a high subgroup of G.

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The study of the group of extensions, pure extensions, neat extensions, pure-high and neat-high extensions of a group A by a group B depends mainly on the groups A and B. Numerous properties of these extensions, in the above mentioned papers were established by using homological methods and by changing the groups A and B. Splitting extensions play an important role in the study of different types of extensions.

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In chapter II, we have discussed the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ by varying the groups A and B from torsion to torsion-free groups. It will be observed that if B is a torsion group then the behaviour of the group $\mathrm{Hext}_p(B,A)$ is almost the same as that of Next(C,A) for any group C. But if A is a torsion-free group and B is a torsion group, then it will be found that the two groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ do not behave in a similar way. For instance, Next(Q/Z,A) is a reduced algebraically compact group while $\mathrm{Hext}_p($ Q/Z,A$) = 0$. Then we have studied the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ keeping A a torsion-free group and imposing no restriction on the group B. It will be observed that the two groups again do not behave in the same fashion. Finally, we have established that if both the groups A and B are torsion groups, the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ have similar properties.

In chapter III, the group Next(B,A) and $\mathrm{Hext}_p(B,A)$ has been studied by varying the groups A and B from cyclic groups of prime order to direct sum of cyclic groups of prime and prime power order. Exact sequences connecting Next to Hom (theorem 3.2) and relating Hom to Ext to Next (theorem 3.4) have been established. Two subgroups of Next(B,A), in case A is a direct sum of cyclic groups of prime power order have been constructed and the properties of these subgroups and their quotient groups have been studied.

In chapter IV, we have concentrated on finding out a group G that is isomorphic to the group of pure-high extensions for suitable groups A and B. Such groups have been named as $Hext_p$-groups. It has been shown that an elementary p-group is a $Hext_p$-group. Also, all these groups whose pure-high extensions by torsion groups are splitting (that is H_p^t - groups) have been studied. Some of the important properties of H_p^t - groups have also been recorded in this chapter.

Chapter V is devoted to studying the pure-high and neat-high extensions. Conditions under which a K-neat- high extension reduces to a K-pure-high extension and a neat extension coincides with a pure extension have been discussed. Splitting conditions for neat and pure-high extensions, condition on A and B under wich Next(B,A) = 0 and $Next_p(B, A) = 0$ have been deduced and discussed.

CHAPTER-I

PRELIMINARIES

The main purpose of this introductory chapter is to recall some of the definitions, results and other necessary bits of information which will be used frequently in the subsequent chapters. This is being done only to fix up the terminology and notations needed in the sequel and to make this work selfcontained as far as possible. The contents of this chapter are not original and most of the definitions and results are from | 4 |,| 6 | and | 8 |.

In section 1,some basic concepts definitions,elementary properties and results, mainly of divisible subgroups, pure subgroups and neat subgroups are given.

In section 2,various types of exact sequences, commutative diagrams, necessary bits of information and results for the groups of extensions and pure extensions, through homological methods are given. The important concepts of algebraically compact and cotorsion groups with their elementary properties may also be found in this section.

In section 3, high, pure-high, neat and neat-high extensions are defined. Some of the known basic results which will be used in the subsequent chapters are stated without proof.

1- Some Elementary Concepts.

Definition(1.1). Groups in which every element has a finite order are called torsion groups, those in which all the elements $\neq 0$ are of infinite orders are called torsion-free groups.

Definition(1.2). A group in which the orders of the elements are powers of the same prime number p is called a primary or p-group.

Proposition(1.3). Every torsion group is a direct sum of p-groups for various $p \in P$.

Proposition(1.4). If B_t is the maximal torsion subgroup (torsion part) of B, then B/B_t is torsion-free.

Definition(1.5). Groups in which every element has square-free order are called elementary.

Proposition(1.6). Every elementary p-group is a direct sum of cyclic groups of the same order p.

Proposition(1.7). A group A is elementary if and only if every subgroup of A is a direct summand.

Definition(1.8). The Frattini subgroup $\phi(G)$ of G is the intersection of all the maximal subgroups of G, that is,

$$\phi(G) = \bigcap_{p \in P} pG$$

Definition(1.9). The first Ulm subgroup G' of G is the intersection of all the subgroups of G, that is ,

$$G' = \bigcap_{n \in N} nG.$$

Definition(1.10). The quotient group G/G' is called the 0th Ulm factor of G and is denoted by G_0.

Definition(1.11). A subgroup E of a group A is called essential if $E \cap B \neq 0$, whenever B is a non-zero subgroup of A.

Proposition(1.12). A is elementary if and only if A is the only essential subgroup in A.

Definition(1.13). By a free abelian group is meant a direct sum of infinite cyclic groups. If these cyclic groups are generated by elements $x_i (i \in I)$, then the free group F will be,

$$F = \bigoplus_{i \in I} \langle x_i \rangle .$$

Definition(1.14). A group G is called divisible if $nG = G$ for all $n \in N$.

It follows from the definition that an epimorphic image of a divisible group is divisible and a direct sum or a direct product of groups is divisible if and only if each component is divisible.Furthermore, if $D_i(i \in I)$ are divisible subgroups of A, then so is their sum ΣD_i.

<u>Proposition</u>(1.15).Every group G has a unique decomposition $G = dG \oplus R$, where dG is the maximal divisible subgroup of G. If $dG = 0$, then the group G is called reduced.

<u>Proposition</u>(1.16).A divisible subgroup D of a group A is a direct summand of A, that is $A = D \oplus C$ for some subgroup C of A.

<u>Proposition</u>(1.17).Every group can be embedded as a subgroup in a divisible group.

<u>Definition</u>(1.18).A minimal divisible group E containing a group A is called the divisible hull(injective hull) of A.

<u>Definition</u>(1.19).A subgroup H of G is called a pure subgroup if the equation $nx = h$,where $h \in H, n \in N$ is solvable in H whenever it is solvable in G,or equivalently, if ,

$$nH = H \cap nG$$

holds for all natural numbers n.

It is easy to verify that every direct summand is a pure subgroup. 0 and A are pure subgroups of A. The torsion part of a mixed group and its p-components are pure subgroups. These fail in general, to be direct summands. It is to be noted that Q and $Z(p^\infty)$ have no pure subgroups. In torsion-free groups, intersection of pure subgroups is again pure. Purity is an inductive property.

Theorem(1.20). Let B, C be subgroups of A such that $C \leq B \leq A$, then we have,

(1) if C is pure in B, and B is pure in A, then C is pure in A,

(2) if B is pure in A, then B/C is pure in A/C,

(3) if C is pure in A, B/C is pure in A/C, then B is pure in A.

Theorem(1.21). If B is a pure subgroup of A such that A/B is a direct sum of cyclic groups, then B is a direct summand of A.

Proposition(1.22). If H is a subgroup of G such that the factor group G/H is torsion-free, then H is pure in G.

Theorem(1.23). For a subgroup B of a group A the following conditions are equivalent,

(1) B is pure in A,

(2) B/nB is a direct summand of A/nB, for every $n > 0$,

(3) if $C \leq B$ is such that B/C is finitely co-generated, then B/C is a direct summand of A/C.

Theorem(1.24). The following are equivalent conditions for a subgroup B of A,

(1) B is pure in A,

(2) P is a direct summand of $n^{-1}B$, for every $n > 0$,

(3) If C is a group between B and A such that C/B is finitely generated, then P is a direct summand of C.

It is to be noted that a group is pure in every group containing it exactly if it is divisible.

Theorem(1.25). If G is a pure subgroup of the group A, then,

(1) $G' = G \cap A'$,

(2) $(G + A')/A'$ is pure in A/A',

(3) $G \leq A'$ implies G is divisible.

Definition(1.26). A subgroup H of a group G is called a neat subgroup if the equation $px = h \in H, p \in P$, is solvable in H whenever it is solvable in G. This is equivalent to the requirement that,

$$pH = H \cap pG ,$$

for all $p \in P$.

Every pure subgroup of a group G is neat, but every neat subgroup of G need not be pure. Neatness is a transitive property, it is an inductive property, that is, the union of an ascending chain of neat subgroups is itself a neat subgroup. In torsion-free groups neatness is equivalent to purity.

Example(1.27). The intersection of two or more neat subgroups of a group is not, in general, neat. Let $G = \{a\} + \{b\}$ where a is of order p^3 while b is of order p. Then the subgroups $A = \{a\}$ and $B = \{(pa + b)\}$ are neat in G. But the subgroup $C = A \cap B = \{p^2a\}$ is not neat in G. It is to be noted that B is not pure in G, since it contains p^2a of height 2.

Theorem(1.28). If A is a subgroup of G, the following statements are equivalent,

(1) A is a neat subgroup of G,

(2) A is maximal disjoint from some subgroup K of G,

(3) if K is a subgroup of G maximal disjoint from A, then A is maximal disjoint from K,

(4) $A \cap pG = pA$, for all $p \in P$.

Proposition(1.29). If N is a neat subgroup of G and either N itself is an elementary p-group or the factor group G/N is elementary, then N is a direct summand of G.

Proposition(1.30). If E is a minimal divisible group containing G, then N is a neat subgroup of G if and only if $N = G \cap D$ where D is a divisible subgroup of E.

Proposition(1.31). If K and L are groups then Hom(K,L) denotes the group of homomorphisms of K into L.

If K is an infinite cyclic group, then $Hom(K,L) \cong L$, and if K is a cyclic group of finite order n then,

$$\mathrm{Hom}(K,L) \cong L[n].$$

<u>Proposition</u>(1.33). In each of the following cases $\mathrm{Hom}(K,L) = 0$,

(1) if K is a torsion group and L is a torsion_free group,

(2) if K is a divisible group and L is a reduced group,

(3) if K is a p-group and L is a q-group, $p \neq q$.

<u>Theorem</u>(1.34). If K and L are groups the following hold,

(1) $\mathrm{Hom}(K,L)$ is a torsion-free group, whenever L is a torsion-free group,

(2) if L is torsion-free and divisible, then $\mathrm{Hom}(K,L)$ is torsion-free and divisible,

(3) if K is divisible, then $\mathrm{Hom}(K,L)$ is torsion-free,

(4) if K is torsion-free and divisible then so is $\mathrm{Hom}(K,L)$,

(5) if K is torsion-free and L is divisible, then $\mathrm{Hom}(K,L)$ is divisible.

<u>Theorem</u>(1.35). There exist natural isomorphisms,

$$\mathrm{Hom}\Big(\bigoplus_{i\in I} K_i, C\Big) \cong \prod_{i\in I} \mathrm{Hom}(K_i, C),$$

$$\mathrm{Hom}\Big(K, \prod_{i\in I} L_i\Big) \cong \prod_{i\in I} \mathrm{Hom}(K, L_i).$$

2-Exact Sequences and Extensions.

<u>Definition</u>(1.36). The sequence ,

$$\cdots\cdots \longrightarrow A_{k-1} \xrightarrow{\varphi_{k-1}} A_k \xrightarrow{\varphi_k} A_{k+1} \longrightarrow \cdots\cdots$$

of groups and homomorphisms is called an exact sequence if the image of each homomorphism is equal to the kernel of the next homomorphism.

If the sequence ,

$$0 \longrightarrow A \xrightarrow{\alpha} B$$

is exact then α is a monomorphism.

If the sequence ,

$$B \xrightarrow{\beta} C \longrightarrow 0$$

is exact then β is an epimorphism.

The exactness of the sequence ,

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is equivalent to the fact that α is an isomorphism.

The exact sequence of the form ,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called a short exact sequence.

The short exact sequence,

$$0 \longrightarrow A \longrightarrow D \longrightarrow D' \longrightarrow 0$$

with D(and hence D') divisible, which exists for every group A, is called an injective resolution of A.

The short exact sequence,

$$0 \longrightarrow H \longrightarrow F \longrightarrow A \longrightarrow 0$$

with F(and hence H) free,which exists for every group A, is called the free or projective resolution of A.

<u>Definition</u>(1.37). If A,B,C,and D are groups and α, β, ν, δ, μ are homomorphisms, then the diagrams,

$$\begin{tikzcd}
 & C \arrow[d, "\alpha"] \\
A \arrow[ur, "\nu"] \arrow[r, "\mu"] \arrow[dr, "\delta"'] & B \arrow[d, "\beta"] \\
 & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\nu"'] & B \arrow[d, "\beta"] \\
C \arrow[r, "\delta"'] & D
\end{tikzcd}$$

are commutative if,

$$\alpha\nu = \mu \ , \ \beta\mu = \delta \text{ and } \beta\alpha = \delta\nu$$

respectively.

<u>The five lemma</u>(1.38).Consider the commutative diagram with exact rows of groups,

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\alpha_1"'] & A_2 \arrow[r] \arrow[d, "\alpha_2"'] & A_3 \arrow[r] \arrow[d, "\alpha_3"'] & A_4 \arrow[r] \arrow[d, "\alpha_4"'] & A_5 \arrow[d, "\alpha_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

(1) if α_2 and α_4 are onto and α_5 is one-to-one, then α_3 is onto.

(2) if α_2 and α_4 are one-to-one and α_1 is onto, then α_3 is one-to-one.

<u>The 3×3 lemma</u>(1.39). Assume that the diagram;

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha_1"] \arrow[d, "\lambda_1"'] & B_1 \arrow[r, "\beta_1"] \arrow[d, "\mu_1"'] & C_1 \arrow[r] \arrow[d, "\nu_1"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\lambda_2"'] & B_2 \arrow[r, "\beta_2"] \arrow[d, "\mu_2"'] & C_2 \arrow[r] \arrow[d, "\nu_2"] & 0 \\
0 \arrow[r] & A_3 \arrow[r, "\alpha_3"] \arrow[d] & B_3 \arrow[r, "\beta_3"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

is commutative and all the three columns are exact.
If the first two or the last two rows are exact, then the remaining row is also exact.

It is to be noted that the lemma can be proved without λ_1 and μ_1 being monomorphisms.

<u>Definition</u>(1.40). An extension $G=(G,f)$ of a group A by a

group B is a pair consisting of a group G and a homomorphism f such that ;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is an exact sequence, where i may stand for the identity mapping.

For any two groups A and B, Ext(B,A) can be defined as equivalence classes of short exact sequence;

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

with addition being Baer addition (see|2|).

<u>Theorem</u>(1.41). If,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence and G is any group, then the following sequences are exact;

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)$$

$$\longrightarrow \mathrm{Ext}(G,A) \longrightarrow \mathrm{Ext}(G,B) \longrightarrow \mathrm{Ext}(G,C) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G)$$

$$\longrightarrow \mathrm{Ext}(C,G) \longrightarrow \mathrm{Ext}(B,G) \longrightarrow \mathrm{Ext}(A,G) \longrightarrow 0 .$$

<u>Theorem</u>(1.42).Let $\{G_i ; i\in I\}$ be any family of groups, then the following isomorphisms hold;

$$\mathrm{Ext}(B, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Ext}(B,G_i) ,$$

$$\mathrm{Ext}(\bigoplus_{i\in I} G_i,A) \cong \prod_{i\in I} \mathrm{Ext}(G_i,A).$$

<u>Definition</u>(1.43).The exact sequence;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is called splitting exact if iA is a direct summend of G, and (G,f) is called a splitting extension of A by B.

Every extension of A by B splits if and only if $\mathrm{Ext}(B,A) = 0$.

<u>Proposition</u>(1.44).In each of the following cases $\mathrm{Ext}(B,A) = 0$;

(1) if A is a divisible group ;

(2) if B is a free group ;

(3) if A is p-divisible and B is a p-group.

<u>Definition</u>(1.45).The exact secuence;

$$0 \longrightarrow A \xrightarrow{i} G \xrightarrow{f} B \longrightarrow 0$$

is called pure exact if iA is a pure subgroup of G and (G,f)

is called a pure extension of A by B.

The pure extensions of A by B form a subgroup Pext(B,A) of Ext(B,A) which coincides with the Ist Ulm subgroup of Ext(B,A), that is :

$$\mathrm{Pext}(B,A) = \bigcap_{n \in N} n\,\mathrm{Ext}(B,A) .$$

<u>Theorem</u>(1.46). If;

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a pure exact sequence, then for any group G the following sequences are exact;

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)$$

$$\longrightarrow \mathrm{Pext}(G,A) \longrightarrow \mathrm{Pext}(G,B) \longrightarrow \mathrm{Pext}(G,C) \longrightarrow 0 ,$$

and

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G)$$

$$\longrightarrow \mathrm{Pext}(C,G) \longrightarrow \mathrm{Pext}(B,G) \longrightarrow \mathrm{Pext}(A,G) \longrightarrow 0.$$

<u>Theorem</u>(1.47). Let $\{G_i \,;\, i \in I\}$ be a family of groups, then the following isomorphisms hold;

$$\mathrm{Pext}(B, \pi_{i \in I} G_i) \cong \pi_{i \in I} \mathrm{Pext}(B,G_i),$$

$$\mathrm{Pext}\Big(\bigoplus_{i\in I} G_i, A\Big) \cong \prod_{i\in I} \mathrm{Pext}(G_i, A).$$

<u>Proposition</u>(1.48). In each of the following cases $\mathrm{Pext}(B,A) = 0$:

(1) for all A, if and only if B is a direct sum of cyclic groups;

(2) for all B, if and only if $A \cong D \oplus S$, where D is a divisible group and S is a direct summand of a direct product of finite cyclic groups ;

(3) if and only if A is algebraically compact for all B.

<u>Definition</u>(1.49). A group G is called algebraically compact if it is a direct summand of every group which contains it as a pure subgroup. This is ecuivalent to the requirement that G is algebraically compact if and only if $\mathrm{Pext}(B,G) = 0$ for all groups B.

Using homological methods it is not difficult to prove that the conditions $\mathrm{Pext}(Q/Z,G) = 0$ and $\mathrm{Ext}(Q,G) = 0$ imply that $\mathrm{Pext}(B,G) = 0$ for all groups B. Thus a group G is algebraically compact if and only if ;

$$\mathrm{Pext}(Q/Z,G) = 0 \text{ and } \mathrm{Ext}(Q,G) = 0.$$

Divisible groups are algebraically compact, a group is algebraically compact if and only if its reduced part is algebraically compact.

Proposition(1.50). Every group can be embedded as a pure subgroup in an algebraically compact group.

Proposition(1.51). If A is algebraically compact, then its first Ulm subgroup coincides with its maximal divisible subgroup.

Proposition(1.52). If A is torsion-free, Ext(C,A) is algebraically compact, whenever C is.

Proposition(1.53). If A is algebraically compact, then Ext(C,A) is a reduced algebraically compact group.

Proposition(1.54). If A is a torsion group, then Hom(A,C) is a reduced algebraically compact group, for any C.

Proposition(1.55). A group G is called cotorsion if all of its extension by torsion-free groups are splitting, that is if Ext(B,G) = 0 for all torsion-free groups B. So a group G is cotorsion if Ext(Q,G) = 0.

Every algebraically compact group is cotorsion, conversely, a torsion-free cotorsion group is algebraically compact.

<u>Theorem</u>(1.56). Every cotorsion group G has a unique decomposition into a direct sum of three groups,

$$G = D \oplus A \oplus B ,$$

where D is a divisible group, A a reduced torsion-free cotorsion group, and B is a reduced cotorsion group having no torsion-free direct summand $\neq 0$.

<u>Proposition</u>(1.57). A group is cotorsion if and only if it is an epimorphic image of an algebraically compact group.

<u>Proposition</u>(1.58). A reduced cotorsion group is algebraically compact if and only if its first Ulm subgroup vanishes.

<u>Proposition</u>(1.59). $\mathrm{Ext}(C,A)$ is cotorsion for all groups A and C.

<u>Propostion</u>(1.60). If G is a cotorsion group, then $\mathrm{Hom}(A,G)$ is cotorsion for any A.

3- High Extensions.

<u>Definition</u>(1.61). Let G be a group and H is a subgroup of G maximal with respect to $H \cap G' = 0$, then H is called a high subgroup of G.

If H is a high subgroup of G, then H is pure in G and G/H is divisible, see | 14 |.

<u>Definition</u>(1.62). If $H' = 0$ and D is a divisible group, then the exact sequence,

$$0 \longrightarrow H \xrightarrow{f} X \longrightarrow D \longrightarrow 0$$

is called a high exact sequence or a high extension of H by D if $f(H)$ is a high subgroup of X.

<u>Definition</u>(1.63). An extension,

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0$$

is an essential extension if M is a subgroup of X and $A \cap M = 0$ implies $M = 0$. The subgroup A is also called an essential subgroup of X.

A homomorphism $f : X \longrightarrow Y$ is an essential homomorphism if Ker f is an essential subgroup of X.

<u>Definition</u>(1.64). Let Shom(X,Y) be the set of essential homomorphisms in Hom(X,Y), then Shom(X,Y) is a subgroup of Hom(X,Y).

<u>Theorem</u>(1.65). Let H be a group with no elements of infinite height (i.e $H' = 0$) and D is a divisible group ,

if ,
$$0 \longrightarrow H \longrightarrow C \longrightarrow C/H \longrightarrow 0$$

is a pure injective resolution of H, then the sequence,

$$\mathrm{Phom}(D,C/H) \longrightarrow \mathrm{Pext}(D,H) \longrightarrow 0$$

is exact, in particular, $\mathrm{Pext}(D,H)$ is a subgroup of $\mathrm{Ext}(D,H)$.

<u>Proposition</u>(1.66).If A is a torsion group and E is a divisible group, then ,

$$\mathrm{Phom}(A,E) = \bigcap_{p\in P} p\,\mathrm{Hom}(A,E).$$

<u>Proposition</u>(1.67). If $H' = 0$ and D is a torsion divisible group then,

$$\mathrm{Pext}(D,H) = \bigcap_{p\in P} p\,\mathrm{Ext}(D,H).$$

<u>Definition</u>(1.68). The exact sequence ,

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0$$

is a pure- high extension if and only if there exists a subgroup K of G such that A is maximal disjoint from K and $(A + K)/K$ is pure in G/K.

<u>Theorem</u>(1.69). Let A and K be subgroups of G, if A is maximal disjoint from K then $(A + K)/K$ is pure in G/K (and hence A is pure-high with respect to K) if and only if for all $n \in N$,

$$K \cap (A + nG) = K \cap nG.$$

<u>Theorem</u>(1.70). Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{i} C \longrightarrow 0$$

be a pure-high exact sequence, then for any group G the following sequences are exact,

$$0 \longrightarrow \mathrm{Hom}(G,A) \longrightarrow \mathrm{Hom}(G,B) \longrightarrow \mathrm{Hom}(G,C)$$

$$\longrightarrow \mathrm{Hext}_p(G,A) \longrightarrow \mathrm{Hext}_p(G,B) \longrightarrow \mathrm{Hext}_p(G,C) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Hom}(C,G) \longrightarrow \mathrm{Hom}(B,G) \longrightarrow \mathrm{Hom}(A,G)$$

$$\longrightarrow \mathrm{Hext}_p(C,G) \longrightarrow \mathrm{Hext}_p(B,G) \longrightarrow \mathrm{Hext}_p(A,G) \longrightarrow 0.$$

<u>Theorem</u>(1.71), If A is a torsion group, then,

$$\mathrm{Hext}_p(A,H) = \bigcap_{p \in P} p\ \mathrm{Pext}(A,H).$$

__Theorem__(1.72). The exact sequence,

$$0 \longrightarrow A \xrightarrow{\ i\ } G \xrightarrow{\ f\ } B \longrightarrow 0$$

is called a neat exact sequence if iA is a neat subgroup of G.

The elements of the group $Next(B,A)$ are the neat exact sequences.

__Theorem__(1.73). If the sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is neat exact, then for any group G the following sequences are exact :

$$0 \longrightarrow Hom(G,A) \longrightarrow Hom(G,B) \longrightarrow Hom(G,C)$$

$$\longrightarrow Next(G,A) \longrightarrow Next(G,B) \longrightarrow Next(G,C) \longrightarrow 0,$$

$$0 \longrightarrow Hom(C,G) \longrightarrow Hom(B,G) \longrightarrow Hom(A,G)$$

$$\longrightarrow Next(C,G) \longrightarrow Next(B,G) \longrightarrow Next(A,G) \longrightarrow 0.$$

__Definition__(1.74). The exact sequence,

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a neat-high extension if there exists a subgroup K of G such that A is maximal disjoint from K and $(A + K)/K$ is neat in G/K.

Theorem(1.75). Let A and K be subgroups of G, such that A is maximal disjoint from K, then $(A + K)/K$ is neat in G/K (and hence A is neat-high with respect to K) if and only if, for all primes p,

$$K \cap (A + pG) = K \cap pG.$$

A result analogous to theorem(1.70) holds good in case of neat-high extensions also,(see[8]).

C H A P T E R-II

GROUPS OF NEAT AND PURE-HIGH EXTENSIONS OF TORSION AND TORSION-FREE GROUPS

Many results for the group of extensions have been proved analogously for the group of pure extensions in | 4 | and | 6 |. The questions that arise are : what happens to the Frattini subgroups of Ext(B,A) and Pext(B,A)? Do they behave in a similar manner? Can the results on Frattini subgroup of Ext be proved analogously for the Frattini subgroup of Pext ? If so, then upto what extent and under what conditions ?

Harrison proved in | 8 | that if B is a torsion group then $\mathrm{Next}_p(B,A)$ is the Frattini subgroup of Pext(B,A). Also,

$$\mathrm{Next}(B,A) = \bigcap_p p\,\mathrm{Ext}(B,A)$$

(see proposition 2.8), that is Next(B,A) is the Frattini subgroup of Ext(B,A).

This chapter aims at studying the groups $Hext_p(B,A)$ and $Next(B,A)$ by varying the groups A and B from torsion to torsion-free groups. Homological methods and techniques are generally employed to develop the results.

Subgroups of cotorsion groups are not in general cotorsion. Also subgroups of algebraically compact groups are not always algebraically compact. In section 1, we prove that the Frattini subgroups of cotorsion groups $Ext(C,A)$ and $Pext(B,A)$, B a torsion group are always cotorsion. Furthermore we establish that the Frattini subgroup $Hext_p(C,A)$ is algebraically compact whenever $Pext(C,A)$ is. A similar result holds good for the Frattini subgroup $Next(C,A)$ of $Ext(C,A)$. The behaviour of the factor groups are almost similar.

In section 2, we establish the natural isomorphisms concerning Next and $Hext_p$ which will frequently be needed in the sequel.

In section 3, the role played by $Next(B,A)$ and $Hext_p(B,A)$, where B is a torsion group and A, a torsion-free group will be discussed. It will be observed that in this case both the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ do not behave in a similar way. Next(Q/Z,A) is a reduced algebraically compact group, while $\mathrm{Hext}_p(Q/Z,A) = 0$, whenever A is a torsion-free group. Furthermore, their behaviours towards isomorphisms will be found different.

In section 4, we shall study the natures of the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$, whenever both the groups B and A are torsion groups. It will be seen that both the groups Next(B,A) and $\mathrm{Hext}_p(B,A)$ behave in a similar way. It is proved that if A_t is the torsion part of A, then $\mathrm{Next}(Q/Z,A_t)$ is an algebraically compact group, whenever Next(Q/Z,A) is. Also, the group $\mathrm{Hext}_p(Q/Z,A_t)$ is an algebraically compact group whenever $\mathrm{Hext}_p(Q/Z,A)$ is.

The results of this chapter are mostly from |19|.

1 - The Group Hext_p .

It is known that the group of all extensions of A by B is (for arbitrary groups A and B) a cotorsion group. Subgroups of cotorsion(algebraically compact) groups are not in general cotorsion (algebraically compact). We investigate in this section those subgroups of cotorsion(algebraically compact) groups which are always cotorsion(algebraically compact).

In this direction we first prove that the first Ulm subgroup of Ext is cotorsion.

<u>Lemma</u>(2.1) Pext(C,A) is for all groups C and A a cotorsion group.

<u>Proof</u>. Since every group A can be embedded as a pure subgroup in a algebraically compact group B, there exists a pure exact sequence ;

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

with B/A algebraically compact (see exercise 3 page 162 of | 4 |). Now, for any group C the sequence ;

$$\mathrm{Hom}(C,B/A) \longrightarrow \mathrm{Pext}(C,A) \longrightarrow \mathrm{Pext}(C,B)$$

is exact. The algebraic compactness of B/A implies the algebraic compactness of Hom(C,B/A) (see theorem 47.7 of | 4 |).

The last group is 0, since P is algebraically compact (see theorem 53.4 of | 4 |)

Pext(C,A) being the epimorphic image of an algebraically compact group and hence cotorsion.(see proposition 54.1 of | 4 |).//

Next, we prove that the group of pure-high extensions is cotorsion.

<u>Theorem</u>(2.2). If C is a torsion group, then the group $\mathrm{Hext}_p(C,A)$ is cotorsion for any group A.

<u>Proof</u>. Since,

$$\mathrm{Hext}_p(C,A) \subseteq \mathrm{Pext}(C,A).$$

The exact sequence;

$$0 \longrightarrow \mathrm{Hext}_p(C,A) \longrightarrow \mathrm{Pext}(C,A) \longrightarrow \mathrm{Pext}(C,A)/\mathrm{Hext}_p(C,A) \longrightarrow 0$$

induces the exact sequence;

$$\mathrm{Hom}(C,\ \mathrm{Pext}(C,A)/\mathrm{Hext}_p(C,A))$$

$$\longrightarrow \mathrm{Ext}(C,\mathrm{Hext}_p(C,A)) \longrightarrow \mathrm{Ext}(C,\mathrm{Pext}(C,A)).$$

Since, C is a torsion group it follows that the factor group (see theorem 55.3 of | 4 |)

$$\mathrm{Pext}(C,A)/\ \mathrm{Hext}_p(C,A)$$

is reduced, while Q is a divisible group, hence the first group is 0 . Also, the last group is 0, because $\mathrm{Pext}(C,A)$ is a cotorsion group (lemma 2.1). Consequently;

$$\mathrm{Ext}(\,Q,\mathrm{Hext}_p(C,A)) = 0,$$

and $\mathrm{Hext}_p(C,A)$ is a cotorsion group.//

The analogue of theorem (2.2) in case of neat extensions is contained in

<u>Proposition</u>(2.3). $\mathrm{Next}(B,A)$ is for all groups A and B a cotorsion group.

<u>Proof</u>. Since,

$$\mathrm{Next}(B,A) \leq \mathrm{Ext}(B,A)$$

the sequence;

$$0 \longrightarrow \mathrm{Next}(B,A) \longrightarrow \mathrm{Ext}(B,A) \longrightarrow \mathrm{Ext}(B,A)/\mathrm{Next}(B,A) \longrightarrow 0$$

is exact. The factor group :

$$\mathrm{Ext}(B,A)/\ \mathrm{Next}(B,A)$$

is reduced. In fact, we even have that its Frattini subgroup is 0. The exact sequence induces the exact sequence

$$\mathrm{Hom}(Q,\mathrm{Ext}(B,A)/\mathrm{Next}(B,A)) \longrightarrow \mathrm{Ext}(Q,\mathrm{Next}(B,A))$$

$$\longrightarrow \mathrm{Ext}(Q,\mathrm{Ext}(B,A)).$$

The first group is 0, since Q is a divisible group and the factor group is a reduced group. Also, $\mathrm{Ext}(B,A)$ is a cotorsion group for all groups A and B (see theorem 54.6 of | 4 |). Hence the last group vanishes. Therefore,

$$\mathrm{Ext}(Q,\mathrm{Next}(B,A)) = 0$$

and $\mathrm{Next}(B,A)$ is a cotorsion group.//

In the next proposition we discuss under what conditions will $\mathrm{Next}_p(C,A)$ be algebraically compact.

__Proposition__(2.4). If C is a torsion group, then $\mathrm{Next}_p(C,A)$ is algebraically compact, whenever $\mathrm{Pext}(C,A)$ is.

__Proof__. Under the given conditions and in view of what we have proved in lemma(2.1) Pext(C,A) is a reduced cotorsion group, which is algebraically compact also. Hence the first Ulm subgroup of Pext(C,A) must vanish (see the remark following corollary 58.2 of | 4 |). But if the first Ulm subgroup of Pext(C,A) vanishes, so does the first Ulm subgroup of $Hext_p(C,A)$.

Thus the first Ulm subgroup of a reduced cotorsión group $Hext_p(C,A)$ is 0. Hence (by proposition 54.2 of | 4 |) $Hext_p(C,A)$ is algebraically compact.//

An analogous result for neat extensions can be proved exactly along the same lines and is contained in

<u>Proposition</u>(2.5). If Ext(C,A) is algebraically compact, so is Next(C,A).//

Concerning the factor group, we prove the following

<u>Proposition</u>(2.6). If C is a torsion group then the factor group;

$$Pext(C,A)/Hext_p(C,A)$$

is algebraically compact for all groups A.

Proof. The Frattini subgroup, and hence the first Ulm subgroup of the factor group;

$$\mathrm{Pext}(C,A)/\mathrm{Next}_p(C,A)$$

is 0. Furthermore, the homomorphic image of a cotorsion group is cotorsion. It follows that the factor group;

$$\mathrm{Pext}(C,A)/\mathrm{Next}_p(C,A)$$

is a reduced cotorsion group whose first Ulm subgroup vanishes. Hence it is algebraically compact.//

In case of Next, the proof of the following proposition is clear.

Proposition(2.7). The factor group;

$$\mathrm{Ext}(C,A)/\mathrm{Next}(C,A)$$

is for all groups A and C an algebraically compact group.//

2- The Group Next.

In this section, we shall prove some important results of the group of neat extensions, which will be frequently used in the sequel. Their analogues for the group of pure-high extensions will be stated. The proofs, which are along similar lines as those of neat extensions are omitted.

First we prove that Next(C,A) is the Frattini subgroup of Ext(C,A).

<u>Proposition</u>(2.8). For any group A and C;

$$\mathrm{Next}(C,A) = \bigcap_p p\mathrm{Ext}(C,A).$$

<u>Proof</u>. The proof of this proposition is an easy consequence of theorem(53.3) of | 4 | . Which states, that an extension G of A by C is divisible by a prime p if and only if A/pA is a direct summand of G/pA.

But, A/pA being a direct summand of G/pA for all p∈P is equivalent to the neatness of A in G.

This proves the proposition.//

Analogue of this proposition for the group of pure-high extension is contained in,

<u>Proposition</u>(2.9). For a torsion group, C ;

$$\mathrm{Hext}_p(C,A) = \bigcap_p p\mathrm{Pext}(C,A).$$

Proof. For proof of this proposition see(theorem 7 of | 8 |).//

In the next theorem we discuss the natural isomorphisms of the groups of neat extensions.

Theorem(2.10). Let $\{G_i ; i\in I\}$ be a family of groups, then for any group H the following isomorphisms hold.

$$\mathrm{Next}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \mathrm{Next}(G_i, H),$$

$$\mathrm{Next}(H, \prod_{i\in I} G_i) \cong \prod_{i\in I} \mathrm{Next}(H, G_i).$$

Proof. Frattini subgroups of two isomorphic groups are isomorphic, and Frattini subgroup of a direct product is the direct product of the Frattini subgroups.

The isomorphism of theorem(52.2) of | 4 |;

$$\mathrm{Ext}(\bigoplus_{i\in I} G_i, H) \cong \prod_{i\in I} \mathrm{Ext}(G_i, H),$$

$$\Longrightarrow \bigcap_{p\in P} p(\mathrm{Ext}(\bigoplus_{i\in I} G_i, H)) \cong \bigcap_{p\in P} p(\prod_{i\in I} \mathrm{Ext}(G_i, H)),$$

$$\Longrightarrow \bigcap_{p\in P} p(\mathrm{Ext}(\bigoplus_{i\in I} G_i, H)) = \prod_{i\in I} \bigcap_{p\in P} p\mathrm{Ext}(G_i, H)),$$

$$\Longrightarrow \operatorname{Next}(\bigoplus_{i\in I} G_i, H) \cong \pi_{i\in I} \operatorname{Next}(G_i, H).$$

see proposition(2.8).

Again, the second isomorphism of theorem(52.2) of | 4 |, that is;

$$\operatorname{Ext}(H, \pi_{i\in I} G_i) \cong \pi_{i\in I} \operatorname{Ext}(H, G_i),$$

$$\Longrightarrow \bigcap_{p\in P} p\operatorname{Ext}(H, \pi_{i\in I} G_i) \cong \bigcap_{p\in P} p(\pi_{i\in I} \operatorname{Ext}(H, G_i)),$$

$$\Longrightarrow \bigcap_{p\in P} p\operatorname{Ext}(H, \pi_{i\in I} G_i) \cong \pi_{i\in I} \bigcap_{p\in P} p\operatorname{Ext}(H, G_i),$$

$$\Longrightarrow \operatorname{Next}(H, \pi_{i\in I} G_i) \cong \pi_{i\in I} \operatorname{Next}(H, G_i).$$

This proves the theorem completely.//

In case of pure-high extensions the following theorem can be proved.

<u>Theorem</u>(2.11). Let $\{G_i ; i\in I\}$ be a family of torsion groups, then for any torsion group H the following isomorphisms holds good,

$$\operatorname{Hext}_p(\bigoplus_{i\in I} G_i, H) \cong \pi_{i\in I} \operatorname{Hext}_p(G_i, H),$$

$$\operatorname{Hext}_p(H, \pi_{i\in I} G_i) \cong \pi_{i\in I} \operatorname{Hext}_p(H, G_i).$$

Proof. Using proposition (2.9), the proof of this proposition is much the same as that of theorem(2.10) and is therefore omitted.//

3- Neat And Pure-High Extensions of A Torsion-free Group.

The aim of this section is to study the neat and pure-high extensions of a torsion-free group A by any group B. First we fix the group $B = Q/Z$, and investigate the behaviour of the groups $Next(B,A)$ and $Hext_p(B,A)$. Next, we study the extensions for any group B. The results will be proved for neat extensions and their analogues for pure-high extensions will be proved if they are not along similar lines.

In the following theorem we prove that $Next(Q/Z,A)$ is an algebraically compact group.

Theorem(2.12). Let D be the divisible hull of any torsion-free group A, then for any monomorphism g such that

$$g : A \xrightarrow{\text{into}} D \oplus \prod_{p \in P} (A/pA)$$

we have,

$$Next(Q/Z,A) \cong Hom(Q/Z,(D \oplus \prod_{p \in P} (A/pA))/gA).$$

Hence,

Next$(Q/Z, A)$ is a reduced algebraically compact group.

<u>Proof</u>. Since, D is the divisible hull of A, the sequence;

$$0 \longrightarrow A \xrightarrow{f} D \longrightarrow D/A \longrightarrow 0$$

is an exact sequence. Define a monomorphism g;

$$g : A \xrightarrow{\text{into}} D \oplus \underset{p \in P}{\pi} (A/pA)$$

in such a way that ,

$$g(a) = (f(a), \ \{a + pA\} \ ), \text{ for } \ a \in A .$$

First we prove that gA is a neat subgroup of

$$D \oplus \underset{p \in P}{\pi} (A/pA).$$

If q is any prime number and $\bar{a} \in D$, $a_p \in A$ and

$$(\ \bar{a}, \{a_p + pA\} \ ) \in D \oplus \underset{p \in P}{\pi} (A/pA),$$

such that;

$$q(\ \bar{a} \ , \ \{a_p + pA\} \ ) = g(a)$$
$$= (f(a), \ \{a + pA\}) \ ,$$

then,

$$q\,\bar{a} = f(a) \,,$$

and

$$q\left\{a_p + pA\right\} = \left\{a + pA\right\} .$$

The last equality emplies that ;

$$a \in qA .$$

Suppose,

$$a = qa' \,, \text{ for } a' \in A.$$

Then,

$$g(a) = (f(a),\ \left\{a + pA\right\}\ ),$$

$$= (f(qa'), \left\{qa' + pA\right\}) \,,$$

$$= q(f(a'), \left\{a' + pA\right\}\ ) \,,$$

$$= q(g(a')) \in q(gA).$$

This proves that gA is a neat subgroup of

$$D \oplus \pi_{p \in P} (A/pA).$$

Therefore, the sequence ;

$$0 \longrightarrow A \xrightarrow{g} D \oplus \pi_{p\in P} (A/pA)$$

$$\longrightarrow (D \oplus \pi_{p\in P} (A/pA))/gA) \longrightarrow 0 \ ,$$

is neat exact and yields the exact sequence ;

$$(1) \qquad \mathrm{Hom}(Q/Z, D \oplus \pi_{p\in P} (A/pA) \longrightarrow \mathrm{Hom}(Q/Z, (D \oplus \pi_{p\in P} (A/pA)/gA)$$

$$\longrightarrow \mathrm{Next}(Q/Z, A) \longrightarrow \mathrm{Next}(Q/Z, D \oplus \pi_{p\in P} (A/pA)).$$

Now,

$$\mathrm{Hom}(Q/Z, D \oplus \pi_{p\in P} (A/pA)) = \mathrm{Hom}(Q/Z, D) \oplus \mathrm{Hom}(Q/Z, \pi_{p\in P} (A/pA)).$$

This first summand is 0, since Q/Z is a torsion group and D, the divisible hull of a torsion-free group is torsion-free.

Furthermore,

$$\mathrm{Hom}(Q/Z, \pi_{p\in P} (A/pA)) \cong \pi_{p\in P} \ \mathrm{Hom}(Q/Z, A/pA) \ ,$$

$$= \pi_{p\in P} \mathrm{Hom}(Q/Z, Z(p)),$$

$$= 0.$$

Since, Q/Z is a divisible group and $Z(p)$, the cyclic group of order p, is a reduced group. Therefore,

$$\mathrm{Hom}(Q/Z, D \oplus \underset{p\in P}{\pi}(A/pA)) = 0.$$

Also the last group in the sequence (1) ;

$$\mathrm{Next}(Q/Z, D \oplus \underset{p\in P}{\pi}(A/pA)) = \mathrm{Next}(Q/Z, D) \oplus \mathrm{Next}(Q/Z, \underset{p\in P}{\pi}(A/pA)).$$

The first summand is 0, since D is a divisible group. Also by theorem(2.10) we have ;

$$\begin{aligned} \mathrm{Next}(Q/Z, \underset{p\in P}{\pi}(A/pA)) &\cong \underset{p\in P}{\pi}\ \mathrm{Next}(Q/Z, A/pA), \\ &= \underset{p\in P}{\pi}\ \mathrm{Next}(Q/Z, Z(p)), \\ &= 0\ . \end{aligned}$$

Since, $Z(p)$ is an elementary p-group see remark(5.7). Hence;

$$\mathrm{Next}(Q/Z, D \oplus \underset{p\in P}{\pi}(A/pA)) = 0.$$

Thus the first and the last groups in the exact sequence (1) vanish , Consequently, we obtain;

$$\text{Next}(Q/Z,A) \cong \text{Hom}(Q/Z,(D \oplus \prod_{p\in P} (A/pA))/gA).$$

Since Q/Z is a torsion group it follows from theorem (46.1) of | 4 | that;

$$\text{Hom}(Q/Z,(D \oplus \prod_{p\in P} (A/pA))/gA),$$

and hence $\text{Next}(Q/Z,A)$ is a reduced algebraically compact group.//

Now focusing on pure-high extensions, we obtain an interesting result, that is : the pure-high extensions of a torsion-free group by Q/Z splits and is contained in,

<u>Theorem</u>(2.13). For a torsion-free group A;

$$\text{Hext}_p(Q/Z,A) = 0.$$

<u>Proof</u>. By proposition(2.9);

$$\text{Hext}_p(Q/Z,A) = \bigcap_{p\in P} p\text{Pext}(Q/Z,A).$$

Since, Q/Z is a torsion group it follows that $\text{Pext}(Q/Z,A)$ is a reduced group.

On the other hand (by property N page 224 of | 4 |) Ext(Q/Z, A) is an algebraically compact group and its first Ulm subgroup Pext(Q/Z, A) must be divisible (see exercise 7 page 162 of | 4 |)

Hence, Pext(Q/Z, A) and therefore

$\mathrm{Next}_p(Q/Z, A) = 0.$ //

Corollary(2.14). For a torsion group C and torsion-free group A; Pext(C, A) = 0.

In the next theorem we examine the groups Next(C, A) and $\mathrm{Next}_p(C, A)$, when C is not restricted to a torsion group, but A is a torsion-free group.

Theorem(2.15). If A is a torsion-free group, then for any group C, the following isomorphism holds good;

$$\mathrm{Next}(C, A) \cong \mathrm{Next}(C_t, A) \oplus \mathrm{Next}(C/C_t, A),$$

where C_t is the torsion part of the group C.

Proof. Since C_t, the torsion part of the group C, is a neat subgroup of C, the following sequence;

$$0 \longrightarrow C_t \longrightarrow C \longrightarrow C/C_t \longrightarrow 0$$

is neat exact and yields the exact sequence;

$$\mathrm{Hom}(C_t,A) \longrightarrow \mathrm{Next}(C/C_t,A) \longrightarrow \mathrm{Next}(C,A)$$

$$\longrightarrow \mathrm{Next}(C_t,A) \longrightarrow 0.$$

The first group is 0, since C_t is a torsion group and A, a torsion-free group. Also, in the second group C/C_t is a torsion-free group and (from example 13 page 226 of | 4 |);

$$p\,\mathrm{Ext}(C/C_t,A) = \mathrm{Ext}(C/C_t,A),$$

for all $p \in P$, Hence from proposition (2.8);

$$\mathrm{Next}(C/C_t,A) = \bigcap_{p \in P} p\,\mathrm{Ext}(C/C_t A),$$

$$= \mathrm{Ext}((C/C_t,A).$$

Therefore, $\mathrm{Next}(C/C_t,A)$ is a divisible group, hence the exact sequence;

$$0 \longrightarrow \mathrm{Next}(C/C_t,A) \rightarrowtail \mathrm{Next}(C,A) \rightarrowtail \mathrm{Next}(C_t,A) \rightarrowtail 0$$

is a splitting exact sequence.

Consequently, we have;

$$\mathrm{Next}(C,A) \cong \mathrm{Next}(C/C_t,A) \oplus \mathrm{Next}(C_t,A).//$$

In case of pure-high extension we have the following analogue of theorem(2.15)

<u>Theorem</u>(2.16). If A is a torsion-free group, then for any group C, the following isomorphism holds good;

$$Hext_p(C,A) \cong Hext_p(C/C_t,A) \text{ ,}$$

where C_t, is the torsion part of the group C.

<u>Proof</u>. Proceeding along the same lines as in theorem (2.15) and inview of theorem(2.13) the required isomorphism holds good.//

For a torsion-free group A we establish the following generalisation of theorem(2.12).

<u>Proposition</u>(2.17). If A is a torsion-free group, then $Next(C,A)$ is an algebraically compact group for all groups C.

<u>Proof</u>. The pure exactness of the sequence;

$$0 \longrightarrow C_t \longrightarrow C \longrightarrow C/C_t \longrightarrow 0$$

induces the exactness of the sequence;

$$Pext(C/C_t,A) \longrightarrow Pext(C,A) \longrightarrow Pext(C_t,A) \longrightarrow 0.$$

The last group is 0,(see corollary 2.14).Also,C/C_t is a torsion-free group and implies $Pext(C/C_t,A)$ and hence $Pext(C,A)$ is a divisible group. Since an epimorphic image of a divisible group is divisible and a divisible group is a direct summand of every group in which it is contained, therefore we have;

$$Ext(C,A) \cong Pext(C,A) \oplus Ext(C,A)/Pext(C,A)$$

The o^{th} Ulm factor of $Ext(C,A)$ is algebraically compact, since Ulm factors of cotorsion groups are algebraically compact (see theorem 54.3 of | 4 |). But a group, is algebraically compact exactly if its reduced part is algebraically compact. It follows that $Ext(C,A)$ is algebraically compact. Proposition (2.5) implies that $Next(C,A)$ is algebraically compact.//

4- Neat And pure-High Extensions Of A Torsion Group.

In this section we shall study the neat and pure-high extensions of a torsion group A by a torsion group B. We fix the torsion group $B = Q/Z$, and investigate the role played by $Next(Q/Z,A)$ and $Hext_p(Q/Z,A)$ for a torsion group A.

In the following theorem we discuss the behaviour of the Frattini subgroup of $\mathrm{Ext}(Q/Z,A)$.

Theorem(2.18). Let G_t be the torsion part of G, then;

$$\mathrm{Next}(Q/Z,G) \cong \mathrm{Next}(Q/Z,G_t) \oplus \mathrm{Next}(Q/Z,G/G_t).$$

Hence, $\mathrm{Next}(Q/Z,G_t)$ is an algebraically compact group, whenever, $\mathrm{Next}(Q/Z,G)$ is.

Proof. Since, the torsion part G_t of the group G is a neat subgroup of G, the sequence,(with the notation $G/G_t = F$)

$$0 \longrightarrow G_t \longrightarrow G \longrightarrow F \longrightarrow 0$$

is neat exact and yields the exact sequence;

$$\mathrm{Hom}(Q/Z,F) \longrightarrow \mathrm{Next}(Q/Z,G_t) \longrightarrow \mathrm{Next}(Q/Z,G)$$

$$\longrightarrow \mathrm{Next}(Q/Z,F) \longrightarrow 0.$$

The first group is 0, since Q/Z is a torsion group and F, a torsion-free group.

Let D be the divisible hull of F, then the sequence;

$$0 \longrightarrow F \xrightarrow{\alpha} D \longrightarrow D/F \longrightarrow 0$$

is exact.

Let,

$$g : F \xrightarrow{into} D \oplus \pi_{p \in P} (F/pF)$$

be a monomorphism defined;

$$g(f_1) = (\alpha(f_1), \ \{f_1 + pF\} \ ) \text{, where } f_1 \in F$$

(just as in the proof of theorem(2.12))

Now, by theorem(2.12) and (example 2 page 43 of | 4 |) we have;

$$Next(Q/Z,F) \cong Hom(Q/Z,(D \oplus \pi_{p \in P} (F/pF))/gF),$$

$$\cong Hom(\bigoplus_{p \in P} Z(p^{\infty}),(D \oplus \pi_{p \in P} (F/pF))/gF),$$

$$\cong \pi_{p \in P} Hom(Z(p^{\infty}),(D \oplus \pi_{p \in P} (F/pF))/gF).$$

But, because of proposition(44.3) of | 4 |, the products are torsion-free, hence Next(Q/Z,F) stays torsion-free.

Furthermore, by proposition(2.3) $Next(Q/Z,G_t)$ is a cotorsion group and so the sequence;

$$0 \longrightarrow Next(Q/Z,G_t) \rightarrowtail Next(Q/Z,G) \longrightarrow Next(Q/Z,F) \longrightarrow 0$$

splits, and we have

$$\mathrm{Next}(Q/Z,G) \cong \mathrm{Next}(Q/Z,G_t) \oplus \mathrm{Next}(Q/Z,G/G_t).$$

Since, every direct summand is a pure subgroup it follows that the splitting sequence is pure exact.

Also, a direct summand of an algebraically compact group is algebraically compact hence it follows that $\mathrm{Next}(Q/Z,G_t)$ is algebraically compact, whenever $\mathrm{Next}(Q/Z,G)$ is algebraically compact.//

In case of pure-high extensions theorem(2.18) takes the following form.

<u>Theorem</u>(2.19). If G_t is the torsion part of G, then;

$$\mathrm{Hext}_p(Q/Z,G) \cong \mathrm{Hext}_p(Q/Z,G_t).$$

<u>Proof</u>. The sequence;

$$0 \longrightarrow G_t \longrightarrow G \longrightarrow G/G_t \longrightarrow 0$$

is pure exact and induces for the group Q/Z the exact sequence;

$$\mathrm{Hom}(Q/Z,G/G_t) \longrightarrow \mathrm{Pext}(Q/Z,G_t) \longrightarrow \mathrm{Pext}(Q/Z,G)$$

$$\longrightarrow \mathrm{Pext}(Q/Z,G/G_t) \longrightarrow 0.$$

Since, Q/Z is a torsion group and G/G_t, a torsion-free group, the first and last groups in the exact sequence are 0, see corollary(2.14) and the sequence

$$0 \longrightarrow \mathrm{Pext}(Q/Z, G_t) \longrightarrow \mathrm{Pext}(Q/Z, G) \longrightarrow 0$$

is exact, that is;

$$\mathrm{Pext}(Q/Z, G) \cong \mathrm{Pext}(Q/Z, G_t) .$$

Since, Frattini subgroups of isomorphic groups are isomorphic it follows inview of proposition(2.9) that

$$\mathrm{Hext}_p(Q/Z, G) \cong \mathrm{Hext}_p(Q/Z, G_t).//$$

Let us investigate the behaviour of $\mathrm{Next}(B, A)$ and $\mathrm{Hext}_p(B, A)$, if A is not restricted to a torsion group but B is a torsion group. In this direction(exercise 11 page 231 of | 4 |) provides that if A is an algebraically compact group and C_t the torsion part of C, then the natural isomorphism;

$$\mathrm{Ext}(C_t, A) \cong \mathrm{Ext}(C, A),$$

holds good.

If we concentrate on neat and pure-high extensions we observe that analogous isomorphisms hold good for a cotorsion group.

<u>Proposition</u>(2.20). For a cotorsion group A, the following isomorphisms hold;

$$\mathrm{Next}(C_t, A) \cong \mathrm{Next}(C, A) \; ,$$

$$\mathrm{Hext}_p(C_t, A) \cong \mathrm{Hext}_p(C, A).$$

<u>Proof</u>. The neat exact sequence;

$$0 \longrightarrow C_t \longrightarrow C \longrightarrow C/C_t \longrightarrow 0$$

induces the exact sequence;

$$\mathrm{Next}(C/C_t, A) \longrightarrow \mathrm{Next}(C, A) \longrightarrow \mathrm{Next}(C_t, A) \longrightarrow 0$$

since, A is a cotorsion group, the first group vanishes and the sequence;

$$0 \longrightarrow \mathrm{Next}(C, A) \longrightarrow \mathrm{Next}(C_t, A) \longrightarrow 0$$

is exact .

The proof of second isomorphism is quite clear.//

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CHAPTER-III

GROUP OF NEAT EXTENSIONS OF CYCLIC GROUPS

Harrison, Fuchs, Rangaswamy, Benabdullah, Hauptfleisch and many others in | 6 |, | 8 |, | 4 |, | 22|, | 25|, | 1 |, | 9 | and | 10| etc. discussed the groups of extensions, pure extensions, neat extensions, high extensions, pure-high extensions and neat-high extensions of a group A by a group B. Many results were established and the behaviour of the above mentioned groups was studied by changing the groups A and B.

We wish to study the neat extensions of a group A by a group B, varying the groups A and B. Restricting the groups A and B to cyclic groups of prime order, direct sum of cyclic groups of prime orders, and direct sum of cyclic groups of prime power orders, we shall investigate the role played by the group of neat extensions.

It follows from remark(5.7) in chapter V, that if A is a cyclic group of prime order p, then Next(B,A) = 0 for all groups B. Also, Next(B,A) = 0 if B is a cyclic group of prime order p for every group A see proposition(5.9) in chapter V. By theorem(2.10) in chapter II it follows that Next(B,A) =0, if B is the direct sum of cyclic groups of prime orders.

In this chapter we shall discuss the characterisation of the group Next(B,A), if A is the direct sum of cyclic groups of prime or prime power order and shall investigate the behaviour of the group Next(B,A), if B is the direct sum of cyclic groups of prime power orders. Throughout this chapter homological mehods are used, exact sequences are established to develop the results.

In section 1, we discuss an exact sequence connecting Next to Hom theorem(3.2), using the divisible part of a group G and the direct sum of cyclic groups of prime orders. Next, we establish an exact sequence relating Hom to Ext to Next using the exact sequence;

$$0 \longrightarrow \bigoplus_{p\in P} Z(p^{n-1}) \xrightarrow{f} \bigoplus_{p\in P} Z(p^n) \xrightarrow{g} \bigoplus_{p\in P} Z(p) \longrightarrow 0.$$

With the help of exact sequences of section 1, we prove in section 2, that there exist two subgroups of the group Next, one contained in the other and discuss the decomposition of the quotient groups obtained. It will be seen in theorem (3.6) that one of the quotient groups is cotorsion.

In section 3, we compute the group Next(B,A), if A is finite and infinite cyclic group and B is a cyclic group of prime power order.

The results of this chapter are from |19| and |21|.

1- Exact Sequences.

Harrison discussed in | 7 | an exact sequence relating Pext and Hom and proved the following theorem.

<u>Theorem</u>(3.1). If A is a group without elements of infinite height, that is $A^1 = 0$ then the sequence ;

$$0 \longrightarrow \mathrm{Pext}(C_o, A) \longrightarrow \mathrm{Pext}(C, A) \longrightarrow \mathrm{Hom}(C^1, \hat{A}/A) \longrightarrow 0$$

is exact. Where C_o is the o^{th} Ulm factor of C, that is : $C_o = C^1/C$ and A is the Z- adic completion of A, such that the sequence;

$$0 \longrightarrow A \longrightarrow \hat{A} \longrightarrow \hat{A}/A \longrightarrow 0$$

is pure exact with $\hat{A}/A$ divisible.

<u>Proof</u>. See Harrison | 7 |.//

Analogous to the exact sequence of theorem(3.1) we prove in this section an exact sequence connecting Next and Hom, with the help of the divisible part of a group G and the direct sum of cyclic groups of prime order.

<u>Theorem</u>(3.2). If D is the divisible part of a group G, then the exact sequence;

$$0 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 0$$

yields the exact sequence;

$$0 \longrightarrow \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \mathrm{Next}(G, \bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0,$$

where Z(p) stands for cyclic group of order p.

<u>Proof</u>. The exact sequence;

$$0 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 0$$

induces the exact sequence(see theorem 44.4 of | 4 |) :

$$0 \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p)).$$

But the last group;

$$\mathrm{Hom}(D, \prod_{p\in P} Z(p)) \cong \prod_{p\in P} \mathrm{Hom}(D, Z(p)) = 0.$$

Since D is a divisible group and Z(p), a cyclic group of prime order, is reduced.

We obtain the exact sequence;

$$(1) \quad 0 \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow 0$$

From theorem(9.2) and exercise 9.14 of | 24 | we know that $\bigoplus_{p\in P} Z(p)$ coincides with the maximal torsion subgroup of $\prod_{p\in P} Z(p)$ and the factor group;

$$\prod_{p\in P} Z(p) / \bigoplus_{p\in P} Z(p)$$

is divisible. It follows from theorem(44.5) of | 4 |

that the sequence;

$$0 \longrightarrow \operatorname{Hom}(G/D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

(2) $$\longrightarrow \operatorname{Hom}(G, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

is exact.

The maximal torsion subgroup $\bigoplus_{p\in P} Z(p)$ of $\prod_{p\in P} Z(p)$ is neat in $\prod_{p\in P} Z(p)$ and therefore the sequence;

$$0 \longrightarrow \bigoplus_{p\in P} Z(p) \longrightarrow \prod_{p\in P} Z(p) \longrightarrow \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p) \longrightarrow 0$$

is neat exact and yield the exact sequences;

$$\operatorname{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \operatorname{Hom}(G/D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \operatorname{Next}(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Next}(G/D, \prod_{p\in P} Z(p)),$$

$$\operatorname{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow \operatorname{Hom}(G, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \operatorname{Next}(G, \bigoplus_{p\in P} Z(p)) \longrightarrow \operatorname{Next}(G, \prod_{p\in P} Z(p)).$$

Since ;

$$\mathrm{Next}(G/D, \prod_{p\in P} Z(p)) \cong \prod_{p\in P} \mathrm{Next}(G/D, Z(p)) = 0.$$

Also,

$$\mathrm{Next}(G, \prod_{p\in P} Z(p)) = 0,$$

because $Z(p)$ is an elementary p- group see remark 5.7 in chapter V.

We obtain the exact sequences;

$$\mathrm{Hom}(G/D, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G/D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$(3) \qquad \longrightarrow \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow 0,$$

and

$$(4)\ \mathrm{Hom}(G, \prod_{p\in P} Z(p)) \longrightarrow \mathrm{Hom}(G, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Next}(G, \bigoplus_{p\in P} Z(p)) \longrightarrow 0.$$

The short exact sequences (1),(2),(3) and (4) yield the following commutative diagram;

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(G/D, \pi_{p \in P} Z(p)) \arrow[r, "f_1"] \arrow[d, "g_1"'] & \mathrm{Hom}(G, \pi_{p \in P} Z(p)) \arrow[r] \arrow[d, "g_2"] & 0 \\
0 \arrow[r] & \mathrm{Hom}(G/D, \pi_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p)) \arrow[r, "f_2"] \arrow[d] & \mathrm{Hom}(G, \pi_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p)) \arrow[r] \arrow[d] & \mathrm{Hom}(D, \pi_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p)) \arrow[r] & 0 \\
 & \mathrm{Next}(G/D, \bigoplus_{p \in P} Z(p)) \arrow[r] \arrow[d] & \mathrm{Next}(G, \bigoplus_{p \in P} Z(p)) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

Since,

$$\mathrm{Next}(G, \bigoplus_{p \in P} Z(p)) \text{ and } \mathrm{Hom}(D, \pi_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p))$$

being epimorphic images of ;

$$\mathrm{Hom}(G, \pi_{p \in P} Z(p) / \bigoplus_{p \in P} Z(p))$$

with Kernels $\mathrm{Im}g_2$ and $\mathrm{Im}f_2$.

Also, we have;

$$\mathrm{Im}g_2 = \mathrm{Im}g_2 f_1 = \mathrm{Im}f_2 g_1 \leq \mathrm{Im}f_2$$

the third row can be extended to ;

$$\longrightarrow \mathrm{Hom}(D, \pi_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

Now the three columns and first two rows in the commutative diagram are exact, it follows by 3×3 lemma that the third row is exact.

We obtain the exact sequence;

$$0 \longrightarrow \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \mathrm{Next}(G, \bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Hom}(D, \pi_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

as desired.//

An analogous exact sequence for pure-high extensions which can be proved along the same lines is as follws.

<u>Theorem</u>(3.3). If D is the divisible part of G the exact sequence;

$$0 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 0$$

induces the exact sequence :

$$0 \longrightarrow \mathrm{Next}_p(G/D, \bigoplus_{p\in P} Z(p)) \longrightarrow \mathrm{Next}_p(G, \bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Hom}(D, \prod_{p\in P} Z(p)/ \bigoplus_{p\in P} Z(p)) \longrightarrow 0.//$$

In the next theorem, we connect Hom to Ext to Next in the form of an exact sequence

Theorem(3.4). The exact sequence;

$$0 \longrightarrow \bigoplus_{p\in P} Z(p^{n-1}) \xrightarrow{f} \bigoplus_{p\in P} Z(p^n) \xrightarrow{g} \bigoplus_{p\in P} Z(p) \longrightarrow 0$$

and homomorphism;

$$f_* : \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1})) \longrightarrow \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^n))$$

imply that

$$\mathrm{Im}\, f_* \leq \mathrm{Next}(G, \bigoplus_{p\in P} Z(p^n))$$

and the sequence;

$$0 \longrightarrow \mathrm{Hom}(G, \bigoplus_{p\in P} Z(p^{n-1})) \longrightarrow \mathrm{Hom}(G, \bigoplus_{p\in P} Z(p^n))$$

$$\longrightarrow \mathrm{Hom}(G, \bigoplus_{p\in P} Z(p))$$

$$\longrightarrow \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1})) \xrightarrow{f_*} \mathrm{Next}(G, \bigoplus_{p\in P} Z(p^{n}))$$

$$\xrightarrow{g_*} \mathrm{Next}(G, \bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

is exact for every group G, where $Z(p^n)$ stands for cyclic group of order p^n, $n > 1$ a fixed integer.

<u>Proof</u>. Let

$$0 \longrightarrow H \longrightarrow F \longrightarrow G \longrightarrow 0$$

be the free resolution of G, with F and hence H free. The following diagram is commutative ;

$$\begin{array}{ccccc}
\mathrm{Hom}(H, \bigoplus_{p\in P} Z(p^{n-1})) & \xrightarrow{h} & \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1})) & \longrightarrow & \mathrm{Ext}(F, \bigoplus_{p\in P} Z(p^{n-1})) \\
\Big\downarrow{\scriptstyle f'_*} & & \Big\downarrow{\scriptstyle f_*} & & \\
\mathrm{Hom}(H, \bigoplus_{p\in P} Z(p^{n})) & \xrightarrow{k} & \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n})) & \longrightarrow & \mathrm{Ext}(F, \bigoplus_{p\in P} Z(p^{n})).
\end{array}$$

Since an extension of a group by a free group is splitting, the last two groups are 0 . Also the Frattini subgroup of a direct sum (direct product) is the direct sum (direct product) of the Frattini subgroups and Frattini subgroups of two isomorphic groups are isomorphic, we have for $n \geq 1$,

$$\bigcap_{p\in P} p\, Z(p^n) = Z(p^{n-1})$$

$$\Longrightarrow \bigcap_{p\in P} p\Big(\bigoplus_{p\in P} Z(p^n)\Big) = \bigoplus_{p\in P} Z(p^{n-1}).$$

The group H is free and hence;

$$\mathrm{Hom}\Big(H, \bigoplus_{p\in P} Z(p^n)\Big) \cong \pi\Big(\bigoplus_{p\in P} Z(p^n)\Big).$$

$$\Longrightarrow \bigcap_{p\in P} p\, \mathrm{Hom}\Big(H, \bigoplus_{p\in P} Z(p^n)\Big) \cong \bigcap_{p\in P} p\, \pi\Big(\bigoplus_{p\in P} Z(p^n)\Big),$$

$$\cong \pi \bigcap_{p\in P} p\Big(\bigoplus_{p\in P} Z(p^n)\Big),$$

$$\cong \pi\Big(\bigoplus_{p\in P} Z(p^{n-1})\Big),$$

$$\cong \mathrm{Hom}\Big(H, \bigoplus_{p\in P} Z(p^{n-1})\Big),$$

$$\Longrightarrow \mathrm{Hom}\Big(H, \bigoplus_{p\in P} Z(p^{n-1})\Big) \leq \mathrm{Hom}\Big(H, \bigoplus_{p\in P} Z(p^n)\Big).$$

Hence f_*' stands for the inclusion mapping, and $\mathrm{Im}f_*'$ is the Frattini subgroup of $\mathrm{Hom}(H, \bigoplus_{p\in P} Z(p^n))$, which is mapped under the homomorphism K into the Frattini subgroup of $\mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^n))$. Also we have;

$$\mathrm{Im}f_* = \mathrm{Im}f_*h = \mathrm{Imk}\, f_*'$$

so $\mathrm{Imk}f'_*$ and hence $\mathrm{Im}f_*$ is mapped into the Frattini subgroup of $\mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^n))$, that is,

$$\mathrm{Im}f_* \leq \mathrm{Next}(G, \bigoplus_{p\in P} Z(p^n)).$$

Since the sequence;

$$\mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1})) \xrightarrow{f_*} \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^n))$$

$$\xrightarrow{g_*} \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p)) \longrightarrow 0$$

is exact, the homomorphism g_* maps Frattini subgroup into Frattini subgroup and hence the sequence;

$$\mathrm{Ext}((G, \bigoplus_{p\in P} Z(p^{n-1})) \xrightarrow{f_*} \mathrm{Next}(G, \bigoplus_{p\in P} Z(p^n)) \xrightarrow{g_*} \mathrm{Next}(G, \bigoplus_{p\in P} Z(p))$$

is exact.

We are required to prove that every E in $Next(G, \bigoplus_{p \in P} Z(p))$ is the image of some E' in $Next(G, \bigoplus_{p \in P} Z(p^n))$. But E' in $Ext(G, \bigoplus_{p \in P} Z(p^n))$ exists such that $g_* E' = E$. Also,

$$Im\ f_* = Ker\ g_* \leq Next(G, \bigoplus_{p \in P} Z(p^n)),$$

It follows from theorem(37.1) of | 1 | that no element not in $Next(G, \bigoplus_{p \in P} Z(p^n))$ can be mapped into the Frattini subgroup of $Im g_*$, and hence

$$E' \in Next(G, \bigoplus_{p \in P} Z(p^n)).//$$

2. Subgroups And Quotient Groups.

With the help of the exact sequences discussed in theorems(3.2) and (3.4) we are now in a position to construct two subgroups of $Next(G, \bigoplus_{p \in P} Z(p^n))$, one contained in the other and discuss the decomposition of the quotient group.

<u>Theorem</u>(3.5).If D isthe divisible part of G, then the exact sequence;

$$0 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 0,$$

and the exact sequence;

$$0 \longrightarrow \bigoplus_{p\in P} Z(p^{n-1}) \longrightarrow \bigoplus_{p\in P} Z(p^{n}) \longrightarrow \bigoplus_{p\in P} Z(p) \longrightarrow 0,$$

where $n > 1$, a fixed integer with homomorphisms;

$$f : \operatorname{Next}(G, \bigoplus_{p\in P} Z(p)) \rightarrowtail \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))$$

$$g : \operatorname{Next}(G, \bigoplus_{p\in P} Z(p^{n})) \rightarrowtail \operatorname{Next}(G, \bigoplus_{p\in P} Z(p))$$

$$h : \operatorname{Ext}((G, \bigoplus_{p\in P} Z(p^{n-1})) \rightarrowtail \operatorname{Next}(G, \bigoplus_{p\in P} Z(p^{n}))$$

$$k : \operatorname{Ext}(G/D, \bigoplus_{p\in P} Z(p^{n-1})) \rightarrowtail \operatorname{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1}))$$

yield the isomorphisms;

$$\operatorname{Next}(G, \bigoplus_{p\in P} Z(p^{n}))/ \operatorname{Ker}fg \cong \operatorname{Hom}(D, \prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p)),$$

$$\operatorname{Ker}fg/\operatorname{Im}hk \cong \operatorname{Next}(G/D, \bigoplus_{p\in P} Z(p)) \oplus \operatorname{Ext}(D, \bigoplus_{p\in D} Z(p^{n-1})).$$

<u>Proof</u>.The exact sequences of theorem(51.3) of | 4 |, theorem(1.73) and theorems(3.2), (3.4) yield the following

commutative diagram;

$$\begin{tikzcd}
 & \mathrm{Ext}(G/D, \bigoplus_{p\in P} Z(p^{n-1})) \arrow[r, "k"] \arrow[d, "t"'] & \mathrm{Ext}(G, \bigoplus_{p\in P} Z(p^{n-1})) \arrow[d, "h"] \arrow[r] & \mathrm{Ext}(D, \bigoplus_{p\in P} Z(p^{n-1})) \arrow[r] & 0 \\
 & \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p^{n})) \arrow[r, "\alpha"] \arrow[d, "d"'] & \mathrm{Next}(G, \bigoplus_{p\in P} Z(p^{n})) \arrow[d, "g"] & & \\
0 \arrow[r] & \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) \arrow[r, "t"] \arrow[d] & \mathrm{Next}(G, \bigoplus_{p\in P} Z(p)) \arrow[r, "f"] \arrow[d] & \mathrm{Hom}(D, \prod_{p\in P} Z(p)/ \bigoplus_{p\in P} Z(p)) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns.

Since the product of two epimorphisms f and g is again an epimorphism it follows that fg is an epimorphism hence;

$$\mathrm{Next}(G, \bigoplus_{p\in P} Z(p^n))/ \mathrm{Ker}fg \cong \mathrm{Hom}(D, \prod_{p\in P} Z(p)/ \bigoplus_{p\in P} Z(p)).$$

The commutative diagram and lemma (8.3) of | 4 |

yield the following exact sequence:

$$\mathrm{Next}(G/D,\bigoplus_{p\in P} Z(p^n))\oplus \mathrm{Ext}(G,\bigoplus_{p\in P} Z(p^{n-1}))\xrightarrow{\nabla(a\oplus h)}\mathrm{Next}(G,\bigoplus_{p\in P} Z(p^n))$$

$$\xrightarrow{fg}\mathrm{Hom}(D,\prod_{p\in P} Z(p)/\bigoplus_{p\in P} Z(p))\longrightarrow 0.$$

It means:

$$\mathrm{Ker}\, fg = \mathrm{Im}\, a + \mathrm{Im}\, h.$$

Evidently,

$$\mathrm{Im}\, hk \leq \mathrm{Im}\, h \cap \mathrm{Im}\, a,$$

if $x \in \mathrm{Im}\, h \cap \mathrm{Im}\, a$, then $x = ay$ for some

$$y \in \mathrm{Next}(G/D,\bigoplus_{p\in P} Z(p^n)),$$

since $x \in \mathrm{Ker}\, g$ it follows that:

$$0 = gx = gay = tdy \Longrightarrow dy = 0,$$

implies,

$$y \in \mathrm{Ker}\, d = \mathrm{Im}\, b \Longrightarrow x \in \mathrm{Im}\, ab = \mathrm{Im}\, hk$$

and hence;

$$\mathrm{Im}\, hk = \mathrm{Im}\, h \cap \mathrm{Im}\, a \;.$$

Furthermore;

$$\mathrm{Im}\, hk \leq \mathrm{Im}\, h = \mathrm{Ker}\, g \leq \mathrm{Ker}\, fg .$$

We obtain two subgroups Im hk and Ker fg of

$\mathrm{Next}(G, \bigoplus_{p\in P} Z(p^n))$ such that Im hk is contained in Ker fg

Now;

$$\mathrm{Ker} fg/\mathrm{Im} hk = \mathrm{Im} a/\mathrm{Im} ab \oplus \mathrm{Im} h/\mathrm{Im} hk .$$

It is clear that;

$$\mathrm{Im}\, a\,/\mathrm{Im} ab \cong \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) \;,$$

and

$$\mathrm{Im}\, h \,/\, \mathrm{Im}\, hk \cong \mathrm{Ext}(D, \bigoplus_{p\in P} Z(p^{n-1})).$$

the desired result follows.//

<u>Remark</u>(3.6).If G/D is a direct sum of free groups and

cyclic groups of prime order, then;

$$\mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p)) = \mathrm{Next}(G/D, \bigoplus_{p\in P} Z(p^n))$$

$$= 0$$

$$= \mathrm{Im}\, hk\ ,$$

and therefore;

$$\mathrm{Ker}\ fg \cong \mathrm{Ext}(D, \bigoplus_{p\in P} Z(p^{n-1})).$$

Theorem(3.7). The quotient group Ker fg / Im hk is cotorsion.

Proof. Since direct sum of two cotorsion groups is cotorsion, the proof follows from theorem(54.6) of | 4 | and from proposition (2.3).//

3. Computation Of Next(P,A).

In section 1 and 2 we have discussed the role of Next(B,A), in case A were the direct sum of cyclic groups of prime or prime power order. In this section we focus our attention on the group of Next(B,A) , if P is the direct sum of cyclic groups of prime or prime power order.

If F is the direct sum of cyclic groups of prime order i,e ,

$$F = \bigoplus_{i \in I} F_i$$

then from proposition (2.10);

$$\begin{aligned} Next(F,A) &= Next(\bigoplus_{i \in I} F_i, A) \\ &\cong \pi_{i \in I}\ Next(F_i, A) \\ &= 0. \end{aligned}$$

If F is the direct sum of cyclic groups of prime power order, then again in view of proposition(2.10) it is sufficient to consider F as a cyclic group of prime power order. In the following propostion we distinguish the case where A is an infinite and finite cyclic group, respectively.

<u>Proposition</u>(3.8).If F is a cyclic group of order p^n (where p is a prime number), $n > 1$ an integer and A is an infinite cyclic group, then;

$$\mathrm{Next}(B,A) = Z(p^{n-1}).$$

<u>Proof</u>. From lemma (52.1,D) of | 4 | it follows that;

$$\mathrm{Ext}(B,A) = A/p^n A .$$

Now, A is an infinite cyclic group therefore;

$$\mathrm{Ext}(B,A) = Z(p^n).$$

By proposition (2.8) it follows that;

$$\mathrm{Next}(B,A) = \bigcap_{p\in P} p\ \mathrm{Ext}(B,A)$$

$$= p\ Z(p^n)$$

$$= Z(p^{n-1}).//$$

If A is a cyclic group of prime order, then

$$\mathrm{Next}(B,A) = 0.$$

We may consider A as a cyclic group of prime power order.

If $A = Z(q^m)$, $B = Z(p^n)$, where $p \neq q$, are prime numbers, and $m,n > 1$, integers then;

$$\mathrm{Next}(B,A) = 0$$

Let therefore $A = Z(p^m)$, $B = Z(p^n)$ where $m,n > 1$ integers. It follows that;

$$\mathrm{Ext}(B,A) = Z(p^m)/\ p^n\ Z(p^m),$$

$$= Z(p^m)/\ Z(p^{m-r}),$$

$$= Z(p^r),$$

where $r = \min.(m,n)$. Hence we have;

$$\mathrm{Next}(B,A) = \bigcap_{p \in P} p\ \mathrm{Ext}(B,A),$$

$$= p\ Z(p^r),$$

$$= Z(p^{r-1}).$$

Hence we have proved the following proposition.

<u>Proposition</u>(3.9). If B is a cyclic group of order p^n and A is a cyclic group of order p^m, where p,q are prime numbers and $m,n > 1$ are integers, then

$$\mathrm{Next}(B,A) = Z(p^{r-1}),$$

where $r = \min.(m,n)$.//

Remark(3.10). According to Hauptfleisch [10] there are $p^r - p^{r-1}$ cyclic extensions of A by P. Since a cyclic p- group contains no non-trivial neat subgroups. It follows that the neat extensions of $Z(p^m)$ by $Z(p^n)$ are exactly those extensions which are non-cyclic.

Remark(3.11). The results of this chapter can be carried out analogously for the group of pure high extensions.

CHAPTER-IV

Hext_p And H_p^t - Groups.

If we wish to find out a group C that may be regarded as the group of homomorphisms of a suitable group A by a suitable group B, we find the answer in (example 1 page 181 of | 4 |) and observe that $C \cong \mathrm{Hom}(Z,C)$. If we think of the same problem for the group of extensions, that is : what are groups that are groups of extensions ? an answer is supplied by (§ 54 property H of | 4 |) which states that every reduced cotorsion group G can be considered as the group of extensions of a suitable group A by a suitable group B. In fact we have the natural isomorphism $G \cong \mathrm{Ext}(Q/Z, G)$.

In this chapter we shall concentrate on pure-high extensions and shall find out groups that are groups of pure-high extensions. Such groups will be called

Hext_p - groups. It will be observed that a group which is direct sum of cyclic groups of the same order p is a Hext_p - group.

Groups all of whose extensions by torsion groups split are divisible groups. Cotorsion groups are defined as groups all of whose extensions by torsion-free groups are splitting. That is a group G is a cotorsion group if Ext(Q,G) = 0. But, Ext(Q,G) = Pext(Q,G), when Q is a torsion-free group. It means that a group G is a cotorsion group if Pext(Q,G) = 0. The question which naturally suggest itself is : what will be the groups all of whose pure extensions by a torsion group split ? This question was answered by Harrison in | 8 |, and he called such groups high injective. According to him, a group G such that G'= 0 is high injective (that is G is a summand of every group in which it is high) if Pext(Q/Z,G) = 0

Now focusing on the Frattini subgroup of Pext the question arises : what are the groups all of whose pure-high extensions by torsion groups are splitting ? We shall discuss this question also in this chapter and shall call such groups H_p^t - groups.

In section 1, some basic lemmas are introduced, which will be needed in the sequel.

In section 2, we shall investigate : what groups G can be represented in the form $G = \mathrm{Hext}_p(B,A)$? (for suitable groups A and B). Hext_p - group will be defined and it will be proved that an elementary p - group is a Hext_p - group.

In section 3, we shall discuss the class of groups all of whose pure-high extensions by torsion groups are splitting. Such groups will be named as H_p^t - groups A necessary and sufficient condition for H_p^t - groups will be discussed.

Section 4, will be devoted to discussing the properties of H_p^t - groups. It will be shown that elementary p - groups and torsion-free groups are H_p^t - groups.

The result of this chapter are mostly from |20|

1. Some Lemmas

First we prove an isomorphism between groups of pure-high extensions.

<u>Lemma</u>(4.1). For an elementary p-group A;

$$\mathrm{Hext}_p\Big(\prod_{q\in P} Z(q), A\Big) = \mathrm{Hext}_p\Big(\bigoplus_{p\in P} Z(q), A\Big).$$

<u>Proof</u>. Since $\bigoplus_{q\in P} Z(q)$ coincides with the maximal torsion subgroup of $\prod_{q\in P} Z(q)$ and the factor group;

$$\prod_{q\in P} Z(q) \Big/ \bigoplus_{q\in P} Z(q)$$

is divisible (see theorem 9.2 and exercise 9.14, respectively of | 24 |)

It is easy to verify that the sequence;

$$0 \longrightarrow \bigoplus_{q\in P} Z(q) \longrightarrow \prod_{q\in P} Z(q) \longrightarrow \prod_{q\in P} Z(q) \Big/ \bigoplus_{q\in P} Z(q) \longrightarrow 0$$

is pure-high exact. Theorem 6 of | 8 | implies that the sequence;

$$\mathrm{Hext}_p\Big(\prod_{q\in P} Z(q) \Big/ \bigoplus_{q\in P} Z(q), A\Big) \longrightarrow \mathrm{Hext}_p\Big(\prod_{q\in P} Z(q), A\Big)$$

$$\longrightarrow \mathrm{Hext}_p\Big(\bigoplus_{q\in P} Z(q), A\Big) \longrightarrow 0$$

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is exact.

Since every elementary p-group is a direct sum of cyclic groups of the same order p and does not contain elements of infinite height, it follows that $A' = 0$. Also,

$$\pi_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q)$$

is divisible and hence (by section 3 of | 8 |) we have;

$$\begin{aligned} \mathrm{Hext}_p\big(\pi_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q), A\big) \\ &= \mathrm{Hext}\big(\pi_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q), A\big), \\ &\leq \mathrm{Pext}\big(\pi_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q), A\big), \\ &\leq \mathrm{Next}\big(\pi_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q), A\big), \\ &= 0 . \end{aligned}$$

Since A is an elementary p-group (see remark 5.7), therefore, the sequence;

$$0 \longrightarrow \mathrm{Next}_p\Big(\prod_{q\in P} Z(q), A\Big) \longrightarrow \mathrm{Next}_p\Big(\bigoplus_{q\in P} Z(q), A\Big) \longrightarrow 0$$

is exact and the required isomorphism follws.//

Now we prove another lemma for elementary p-groups.

Lemma(4.2). If A is an elementary p-group, then;

$$\mathrm{Hom}\Big(Q \oplus \prod_{q\in P} Z(q), A\Big) = 0$$

Proof. Since;

$$\mathrm{Hom}\Big(Q \oplus \prod_{q\in P} Z(q), A\Big) \cong \mathrm{Hom}(Q, A) \oplus \mathrm{Hom}\Big(\prod_{q\in P} Z(q), A\Big).$$

Now Q is a divisible group and A is a reduced group it therefore follows that;

$$\mathrm{Hom}(Q, A) = 0.$$

Furthermore, the exactness of the sequence;

$$0 \longrightarrow \bigoplus_{q\in P} Z(q) \longrightarrow \prod_{q\in P} Z(q) \longrightarrow \prod_{q\in P} Z(q)/\bigoplus_{q\in P} Z(q) \longrightarrow 0$$

implies the exactness of the sequence;

$$0 \longrightarrow \mathrm{Hom}\Big(\prod_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q), A \Big) \longrightarrow \mathrm{Hom}\Big(\prod_{q \in P} Z(q), A \Big)$$

$$\longrightarrow \mathrm{Hom}\Big(\bigoplus_{q \in P} Z(q), A \Big)$$

(see theorem(44.4) of | 4 |). The first group is 0, since;

$$\prod_{q \in P} Z(q) / \bigoplus_{q \in P} Z(q)$$

is a divisible group and A , a reduced group. Also the last group in the sequence is 0 since $\bigoplus_{q \in P} Z(q)$ is a q-group and A is a p-group and hence;

$$\mathrm{Hom}\Big(\prod_{q \in P} Z(q), A \Big) = 0. //$$

Finally, we prove another lemma for elementary p-groups.

__Lemma(4.3)__. If A is an elementary p-group, then;

$$\mathrm{Hext}_p\Big(Q \oplus \prod_{q \in P} Z(q), A \Big) = 0.$$

__Proof__. Since;

$$\mathrm{Hext}_p\Big(Q \oplus \prod_{q \in P} Z(q), A \Big) \cong \mathrm{Hext}_p(Q, A) \oplus \mathrm{Hext}_p\Big(\prod_{q \in P} Z(q), A \Big)$$

and $A' = 0$ and Q is a divisible group, it follows from $| 8 |$ that;

$$\begin{aligned}
\mathrm{Hext}_p(Q,A) &= \mathrm{Hext}(Q,A) \\
&\leq \mathrm{Pext}(Q,A) \\
&\leq \mathrm{Next}(Q,A) \\
&= 0,
\end{aligned}$$

since A is an elementary p-group. Also by theorem(2.11);

$$\mathrm{Hext}_p\Big(\bigoplus_{q\in P} Z(q), A\Big) \cong \prod_{q\in P} \mathrm{Hext}_p(Z(q),A).$$

Now, (by proposition 2.9) since $Z(q)$ is a torsion group we have;

$$\begin{aligned}
\prod_{q\in P} \mathrm{Hext}_p(Z(q),A) &= \prod_{q\in P}\Big(\bigcap_{q\in P} q\,\mathrm{Pext}(Z(q),A)\Big), \\
&\leq \prod_{q\in P}\Big(\bigcap_{q\in P} q\,\mathrm{Next}(Z(q),A)\Big), \\
&= 0.
\end{aligned}$$

Consequently, by lemma(4.1) we have

$$\mathrm{Hext}_p(Q \oplus \sum_{q \in P} Z(q), A) = 0.//$$

2. Hext_p - Groups.

The aim of this section is to find out group G that is isomorphic to the group of pure-high extensions for suitable groups A and B. We shall call such groups Hext_p- groups.

Definition(4.4). A group G is called a Hext_p- group if,

$$G \cong \mathrm{Hext}_p(B, A)$$

for suitable groups A and B.

Following is the main theorem of this chapter

Theorem(4.5). Every elementary p-group is the Hext_p-group.

Proof. Imbed Z, the additive group of integers in the divisible group Q, the additive groups of rational numbers, that is

$$0 \longrightarrow Z \xrightarrow{\alpha} Q.$$

Define a monomorphism $\bar{\alpha}$;

$$\bar{\alpha} : Z \longrightarrow Q \oplus \pi_{q \in P} (Z/qZ)$$

such that;

$$\bar{\alpha}\, z = (\alpha z\, ,\ (\ldots\ldots,\ z + qZ,\ \ldots\ldots))$$

where $z \in Z$. Now proceeding as we do in theorem(2.12) in chapter II we obtain that $\bar{\alpha} Z$ is a neat subgroup of ;

$$Q \oplus \pi_{q \in P} (Z/qZ).$$

Now;

$$Q \oplus \pi_{q \in P} (Z/qZ) = Q \oplus \pi_{q \in P} Z(q),$$

where $Z(q)$ stands for cyclic group of order q and $\bar{\alpha} Z$ is maximal disjoint from $\oplus_{q \in P} Z(q)$. This is in accordance with the remark of Harrison | 8 | section 4, that A is a neat subgroup of G if and only if A is maximal disjoint from some subgroup K of G.

It is easy to see that $\bar{\alpha} Z$ is a pure subgroup of $Q \oplus \pi_{q \in P} Z(q)$. Furthermore, $\oplus_{q \in P} Z(q)$ is a maximal torsion subgroup of $\pi_{q\in P} Z(q)$ and therefore of $Q \oplus \pi_{q\in P} Z(q)$. Hence;

$$\bar{\alpha} Z + \bigoplus_{q\in P} Z(q)$$

is a pure subgroup of

$$Q \oplus \pi_{q\in P} Z(q) .$$

(see exercise 5, page 116 of | 4 |). Now by (lemma 26.1 of | 4 |) we have;

$$\bar{\alpha} Z + \bigoplus_{q\in P} Z(q))/ \bigoplus_{q\in P} Z(q)$$

is a pure subgroup of ;

$$(Q \oplus \pi_{q\in P} Z(q))/ \bigoplus_{q\in P} Z(q) .$$

From theorem 4 of | 8 |, it follows that the sequence;

$$0 \longrightarrow Z \xrightarrow{\bar{\bar{x}}} Q \oplus \pi_{q\in P} Z(q)$$

$$\longrightarrow (Q \oplus \pi_{q\in P} Z(q))/ \bar{\alpha} Z \longrightarrow 0$$

is pure-high exact

Theorem 6 of | 8 | implies that for an elementary p-group G the sequence;

$$\mathrm{Hom}(Q \oplus \pi_{q \in P} Z(q), G) \longrightarrow \mathrm{Hom}(Z, G)$$

$$\longrightarrow \mathrm{Hext}_p(Q \oplus \pi_{q \in P} Z(q))/\bar{\alpha}Z, G)$$

$$\longrightarrow \mathrm{Hext}_p(Q \oplus \pi_{q \in P} Z(q), G)$$

is exact.

The first group is 0 by lemma (4.2) and the last group is 0 by lemma (4.3). Also,

$$\mathrm{Hom}(Z, G) \cong G$$

therefore we have;

$$G \cong \mathrm{Hext}_p((Q \oplus \pi_{q \in P} Z(q))/\bar{\alpha}Z, G).$$

This proves that a group which is a direct sum of cyclic groups of the same order p is a Hext_p- group.

3. H_p^t - Groups.

In this section we shall discuss the class of groups all of whose pure-high extensions by torsion groups are splitting.

<u>Definition</u>(4.6). A group is called a H_p^t - group if all its pure-high extensions by a torsion group are splitting. That is a group G is called H_p^t- group if for all torsion groups T;

$$\mathrm{Hext}_p(T,G) = 0$$

We discuss in the following theorem a necessary and sufficient condition for a group G to be a H_p^t - group.

<u>Theorem</u>(4.7). A necessary and sufficient condition for a group G to be a H_p^t - group is that;

$$\mathrm{Hext}_p(Z(p^\infty),G) = 0,$$

for all prime numbers p.

<u>Proof</u>. Only sufficiency needs verification.

Since a torsion group is the direct sum of p-groups, it is sufficient to prove the theorem for p-groups. If T is any p-group, then we have the existence of a pure-high exact sequence;

$$0 \longrightarrow H \longrightarrow T \longrightarrow T/H \longrightarrow 0,$$

with H direct sum of cyclic groups and

$$T/H = \bigoplus Z(p^{\infty}) .$$

The pure-high exact sequence yields the exact sequence

$$\mathrm{Hext}_p(T/H,G) \longrightarrow \mathrm{Hext}_p(T,G) \longrightarrow \mathrm{Hext}_p(H,G)$$

for any group G. The group H is torsion; hence,

$$\mathrm{Hext}_p(H,G) = \bigcap_{p \in P} p\ \mathrm{Pext}(H,G)$$

$$= 0,$$

for H is the direct sum of cyclic groups. Also;

$$\mathrm{Hext}_p(T/H,G) = \mathrm{Hext}_p(\bigoplus_{p \in P} Z(p^{\infty}),G)$$

$$= \pi_{p \in P} \operatorname{Hext}_p(Z(p^{\infty}), G)$$

$$= 0.$$

Consequently,

$$\operatorname{Hext}_p(T, G) = 0. //$$

It is easy to verify that divisible groups and algebraically compact groups are H_p^t - groups. Also it follows immediately from theorem(2.13) that torsion-free groups are H_p^t - groups.

The following proposition asserts that for a group to be a H_p^t - group it is enough to check its reduced part.

<u>Propostion</u>(4.8). A group is H_p^t - group exactly, if its reduced part is.

<u>Proof</u>. Let,

$$G = D \oplus L$$

be the decomposition of the group G into its divisible part D and the reduced part L, then by theorem(2.11) we have;

$$\operatorname{Hext}_p(Z(p^{\infty}), G) = \operatorname{Hext}_p(Z(p^{\infty}), D \oplus L)$$

$$= Hext_p(Z(p^\infty),D) \oplus Hext_p(Z(p^\infty),L)$$

$$= Hext_p(Z(p^\infty),L).//$$

Towards direct product we have the following,

<u>Proposition</u>(4.9). A direct product $\pi_{i \in I} G_i$ is a H_p^t - group if and only if each G_i is .

<u>Proof</u>. The proof follows from the fact that;

$$Hext_p(Z(p^\infty),\pi_{i\in I} G_i) \cong \pi_{i\in I} Hext_p(Z(p^\infty),G_i)$$

$$= 0.//$$

The following theorem gives more insight into H_p^t - groups.

<u>Theorem</u>(4.10). Let

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

be a pure-high exact sequence then the following hold.

(1) If both H and G/H are H_p^t - groups, then so is G.

(2) If G is a H_p^t - group, then so is H, whenever the factor group is reduced.

(3) G is a H_p^t - group if and only if H is a H_p^t - group, whenever G/H is torsion-free.

<u>Proof</u>. The pure-high exact sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

yields the exact sequence;

$$\mathrm{Hext}_p(Z(p^\infty),H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty),G) \longrightarrow \mathrm{Hext}_p(Z(p^\infty),G/H)$$

The first and the last groups are 0, since H and G/H are H_p^t - groups. Therefore,

$$\mathrm{Hext}_p(Z(p^\infty),G) = 0$$

and G is H_p^t - group.

Again the sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

induces the exact sequence

$$\mathrm{Hom}(Z(p^\infty),G/H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty),H) \longrightarrow \mathrm{Hext}_p(Z(p^\infty),G).$$

The first group is 0, since $Z(p^\infty)$ is a divisible group, G/H is a reduced group. Also, the last group is 0, since G is a H_p^t - group. Hence

$$\operatorname{Hext}_p(Z(p^\infty),H) = 0;$$

and (2) follows.

Finally, the sequence:

$$0 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 0$$

gives the exact sequence:

$$\operatorname{Hom}(Z(p^\infty),G/H) \longrightarrow \operatorname{Hext}_p(Z(p^\infty),H) \longrightarrow \operatorname{Hext}_p(Z(p^\infty),G)$$

$$\longrightarrow \operatorname{Hext}_p(Z(p^\infty),G/H).$$

If G is a H_p^t - group, then

$$\operatorname{Hext}_p(Z(p^\infty),G) = 0.$$

Also, G/H is a torsion-free group and $Z(p^\infty)$ is a torsion group, hence

$$\operatorname{Hom}(Z(p^\infty),G/H) = 0$$

therefore,

$$\mathrm{Next}_p(Z(p^{\infty}),H) = 0$$

and H is a H_p^t - group.

If H is a H_p^t - group and G/H is a torsion-free group then the 2nd and 4th groups in the above exact sequence are 0, and G is a H_p^t - group.//

Now we discuss under what conditions a reduced group will be a H_p^t - group.

<u>Theorem</u>(4.11). A reduced group A is H_p^t - group if and only if $\mathrm{Ext}(Q/Z,A)$ is a H_p^t - group.

<u>Proof</u>. The exact sequence;

$$0 \longrightarrow Z \longrightarrow Q \longrightarrow Q/Z \longrightarrow 0$$

induces the exact secuence;

$$\mathrm{Hom}(Q,A) \longrightarrow \mathrm{Hom}(Z,A) \longrightarrow \mathrm{Ext}(Q/Z,A) \longrightarrow \mathrm{Ext}(Q,A)$$

$$\longrightarrow \mathrm{Ext}(Z,A) \longrightarrow 0$$

The first group is 0, also the last group is 0 since Z is a free group. Furthermore $Ext(Q,A)$ is a torsion-free group and

$$Hom(Z,A) \cong A .$$

If follows that the sequence;

$$0 \longrightarrow A \longrightarrow Ext(Q/Z,A) \longrightarrow Ext(Q,A) \longrightarrow 0$$

is pure exact and yields the exact sequece;

$$Hom(Z(p^{\infty}),Ext(Q,A)) \longrightarrow Pext(Z(p^{\infty}),A) \longrightarrow Pext(Z(p^{\infty}),Ext(Q/Z,A))$$

$$\longrightarrow Pext(Z(p^{\infty}),Ext(Q,A)) \longrightarrow 0$$

The first and last groups of the exact sequence are 0. Also, Frattini subgroups of two isomorphic groups are isomorphic it follows that;

$$Hext_p(Z(p^{\infty}),A) \cong Hext_p(Z(p^{\infty}),Ext(Q/Z,A)).//$$

To have more insight into H_p^t - groups we discuss an other characterisation of H_p^t - group, that is :

<u>Theorem</u>(4.12). A is H_p^t - group, if and only if $Pext(Z(p^{\infty}),A)$ is isomorphic to a subgroup of $\pi_{p\in P} Z(p)$.

Proof. Let A be a H_p^t - group, then it follows that;

$$\mathrm{Hext}_p(Z(p^\infty),A) = 0.$$

Consider the homomorphism;

$$\alpha : \mathrm{Pext}(Z(p^\infty),A) \longrightarrow \prod_{p\in P} (\mathrm{Pext}(Z(p^\infty),A)/p\,\mathrm{Pext}(Z(p^\infty),A))$$

defined by;

$$\alpha E = (\,\ldots\ldots\ldots, E + p\,\mathrm{Pext}(Z(p^\infty),A),\ldots\ldots\ldots),$$

where $E \in \mathrm{Pext}(Z(p^\infty),A)$.

Now if $\alpha E = 0$ it follows that

$$(\ldots\ldots\ldots, E + p\,\mathrm{Pext}(Z(p^\infty),A),\ldots\ldots\ldots) = 0,$$

which implies that

$$E \in \bigcap_{p\in P} p\,\mathrm{Pext}(Z(p^\infty),A).$$

Now by proposition (2.9) it follows that

$$E \in \mathrm{Hext}_p(Z(p^\infty),A) = 0$$

Therefore, α is a monomorphism.

Furthermore, since p $\mathrm{Pext}(Z(p^{\infty}),A)$ is the maximal subgroup of $\mathrm{Pext}(Z(p^{\infty}),A)$, we have;

$$\mathrm{Pext}(Z(p^{\infty}),A)/p\,\mathrm{Pext}(Z(p^{\infty}),A) \cong Z(p).$$

Conversely, let

$\mathrm{Pext}(Z(p^{\infty}),A) \cong$ to a subgroup of $\prod_{p\in P} Z(p)$ then,

Frattini subgroups of isomorphic groups will be isomorphic, it follows that;

$$\mathrm{Hext}_p(Z(p^{\infty}),A) = 0,$$

and A becomes a H_p^t - group.//

CHAPTER-V

PURE - HIGH AND NEAT - HIGH EXTENSIONS

Every pure subgroup is a neat subgroup, but the converse is not always true. If $G = \{a\} \oplus \{b\}$, where a and b are of orders p^3 and p, respectively, $p \in P$, then the subgroup $\{pa + b\}$ is neat but not pure in G, see Honda |11|.

Some mathematicians like Kertesz, Szele and Fuchs have tried to classify the groups in which both the concepts coincide. Recently, Simauti have classified the abelian groups in which every neat subgroup is pure.

Turning to extensions it is clear from definitions of pure and neat extensions that every pure extension is a neat extension, but every neat extension need not be a pure extension.

If K is a subgroup of G, then the exact sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

is a K-pure-high extension if H is maximal disjoint from K and $(H + K)/K$ is pure in G/K, and a K-neat-high extension if H is maximal disjoint from K and $(H + K)/K$ is neat in G/K.

Since $(H + K)/K$ is pure in G/K implies $(H + K)/K$ is neat in G/K it follows immediately that every K-pure-high extension is a K-neat-high extension but a K-neat-high extension need not be a K-pure-high extension.

In this chapter we shall discuss ~~some~~ the very natural and interesting questions : under what conditions a K-neat-high extension is a K-pure-high extension and a neat extension is a pure extension?

The results of this chapter are yet to appear in Tamkang Journal of Mathematics |18|.

1. Neat - High Extensions.

In this section, we investigate the conditions under which a K-neat-high extension becomes a K-pure-high extension. We establish in this direction the following lemma.

<u>Lemma</u>(5.1). If n is a square-free natural number, and H and K are subgroups of G, such that(H + K) is a direct sum of cyclic groups of the same order n, then $(H + K)/K$ is a neat subgroup of G/K if and only if $(H + K)/K$ is a direct summand of G/K.

<u>Proof</u>. If$(H+K)/K$ is a neat subgroup of G/K, then it follows for all $p \in P$;

$$p((H + K)/K) = ((H + K)/K) \cap p(G/K)$$

which in turn is equivalent to;

$$m((H + K)/K) = ((H + K)/K) \cap m(G/K)$$

for all square-free natural numbers m.

In particular if $m = n$ we must have;

$$n((H + K)/K) = K,$$

and hence;

$$((H + K)/K) \cap n(G/K) = K.$$

The set of all subgroups of G/K containing $n(G/K)$

and disjoint from $(H + K)/K$ is not empty and is inductive. Hence by Zorn's lemma there exists a maximal member say B/K. Thus B/K is a $(H + K)/K$-high subgroup of G/K, see |13|.

Let ,

$$n = p_1\, p_2\, p_3 \,\cdots\cdots\, p_r \,\cdots\cdots\, p_n$$

be the factorisation of n into different primes. Let

$$g + K \in G/K$$

then;

$$n(g + K) \in n(G/K) \leq B/K$$

and therefore,

$$p_1\, p_2\, p_3 \,\cdots\cdots\, p_r \cdots\cdots p_n(g+K) \in B/K.$$

It follows by lemma (9.8) of | 4 | that

$$p_2\, p_3 \cdots\cdots p_r \cdots\cdots p_n(g+K) \in (H + K)/_K \oplus B/K.$$

The numbers;

$$p_2\, p_3 \cdots\cdots p_r \cdots\cdots p_{n-1}\, p_n \,,$$

$$p_1 \; p_3 \cdots\cdots \; p_r \cdots\cdots \; p_{n-1} \; p_n \; ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$p_1 \; p_2 \cdots\cdots p_r \cdots\cdots\cdots \; p_{n-1}$$

are relatively prime, hence there exist integers

$$\alpha_1 , \alpha_2 \cdots\cdots\cdots\cdots , \alpha_r , \cdots\cdots\cdots , \alpha_n$$

such that

$$1 = \alpha_1 \; p_2 \; p_3 \cdots\cdots \; p_r \cdots\cdots \; p_{n-1} \; p_n$$

$$+ \; p_1 \; \alpha_2 \; p_3 \cdots\cdots \; p_r \cdots\cdots \; p_{n-1} \; p_n$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \; p_1 \; p_2 \cdots\cdots \; p_r \cdots\cdots \; p_{n-1} \; \alpha_n$$

Now for $(g + K) \in G/K$, we must have;

$$(g + K) = \alpha_1 \; p_2 \; p_3 \cdots \; p_r \cdots\cdots \; p_{n-1} \; p_n (g + K)$$

$$+ \; p_1 \; \alpha_2 \; p_3 \cdots\cdots \; p_r \cdots\cdots \; p_{n-1} \; p_n (g + K)$$

$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$$+ p_1 p_2 \cdots\cdots p_r \cdots\cdots p_{n-1} q_n(g + K).$$

Since all the sums in the above equation belong to;

$$(H + K)/K \oplus B/K ,$$

it follows that,

$$G/K \leq (H + K)/K \oplus B/K.$$

The reverse inclusion follows from lemma(9.8) of | 4 | and hence;

$$G/K = (H + K)/K \oplus B/K.$$

It follows that $(H + K)/K$ is a direct summand of G/K. Since every direct summand is a neat subgroup,(see page 91 of | 3 |).

The lemma is completely proved.//

In the next theorem we give the condition under which a K-neat-high extension becomes a K-pure-high extension.

__Theorem__(5.2). If n is a square-free natural number, a K-neat-high extension;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

is a K-pure-high extension if $(H + K)$ is the direct sum of cyclic groups of the same order n.

__Proof__. Since the sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

is a K-neat-high exact, it follows that H is maximal disjoint from K and $(H + K)/K$ is neat in G/K. Furthermore, $(H + K)$ is the direct sum of cyclic groups of the same order n.

The requirements of lemma (5.1) are satisfied and so $(H + K)/K$ is a direct summand of G/K. But every direct summand is a pure subgroup (property 'a' page 114 of | 4 |, it follows that $(H + K)/K$ is a pure subgroup of G/K.

Hence the sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

is K-pure-high exact.//

In case of elementary p-groups we have the following interesting theorem.

Theorem(5.3). A K-neat-high extension;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

is a K-pure-high extension if $(H + K)$ is an elementary p-group.

Proof. An elementary p-group is the direct sum of cyclic groups of the same order p. It follows that $(H + K)$ is the direct sum of cyclic groups of the same order p. Lemma (5.1) completes the proof of the theorem.//

2. Neat And Pure Extensions.

In this section we shall discuss : under what conditions is a neat subgroup a pure subgroup ? Or in other words, under what condi*ti*ons does a neat extension reduce itself to a pure extension ? The following lemma gives an answer to this question.

Lemma(5.4). If n is a square-free natural number and H, a subgroup of a group G, such that H is a direct sum of cyclic groups of the same order n, then H is a neat subgroup of G, if and only if H is a direct summand of G.

Proof. The proof is much the same as that of lemma (5.1).//

Lemma(5.4). Suggests the conditions under which a neat extension is a pure extension and is contained in

Theorem(5.5). If n is a square-free natural number, then the neat extension;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

splits, if H is a direct sum of cyclic groups of the same order n.

Proof. The exactness of the sequence implies that H is a neat subgroup of G. Also H is a direct sum of cyclic groups of the same order n. The requirements of lemma (5.4) are satisfied and therefore H is a direct summand of G.

Hence the sequence;

$$0 \longrightarrow H \longrightarrow G \longrightarrow B \longrightarrow 0$$

splits.

Since every direct summand is pure, the sequence is pure exact as well.//

Concerning the elementary p-groups theorem(5.5) takes the following elegant form.//

Theorem(5.6). The neat extension;

$$0 \longrightarrow H \longrightarrow G \longrightarrow F \longrightarrow 0$$

splits, if H is an elementary p-group.//

Remark(5.7). It follows immediately that for any group F and for an elementary p-group (cyclic group of prime order) H we have;

$$\mathrm{Next}(F,H) = 0.$$

For pure-high extensions we have the following,

Proposition(5.8). For an elementary p-group A and a torsion group F;

$$\mathrm{Hext}_p(F,A) = 0.$$

Proof. From proposition(2.9) it follows that

$$\begin{aligned}\mathrm{Hext}_p(F,A) &\leq \mathrm{Pext}(F,A)\\ &\leq \mathrm{Next}(F,A)\\ &= 0.//\end{aligned}$$

If F is an elementary p-group we have the following,

<u>Proposition</u>(5.9). For an elementary p-group B ;

$$\text{Next}(B,A) = 0.$$

<u>Proof</u>. This is in accordance with the result of Hauptfleisch | 9 | who proved that : If

$$B = Z(n) = \{a\} \quad ,$$

then $f \in T(B,A)$ if and only if ;

$$\sum_{i=1}^{p-1} f(a, ia) \in n A.$$

Consequently, if B is a cyclic group of prime order p, then,

$$F'(B,A) = T(B,A) \text{ so that } \text{Next}(B,A) = 0.$$

For pure-high extensions we have

<u>Proposition</u>(5.10). For an elementary p-group B,

$$\text{Next}_p(B,A) = 0.//$$

BIBLIOGRAPHY

|1| Benabdullah,K.and Irwin,J. : On N-high subgroups of abelian groups.Bull.Soc.Maths.France 96,(1968),337-346.

|2| Cartan,H.,and Eilenberg,s. : Homological algebra Princeton, (1956).

|3| Fuchs, L. : Abelian groups,Publishing house of the Hungarian academy of sciences. Budapest, (1958).

|4| ———————. : Infinite abelian groups Vol.1, Academic Press, New York and London(1970).

|5| ———————. : Notes on abelian groups 11,Acta Math.Acad.Science Hungar,11, (1960),117-125.

|6| Harrison,D.K. : Infinite abelian groups and homological methods. Ann. of Maths.,19,(1959),366-391.

|7| ———————. : On the structure of Ext,topic in abelian groups. Chicago Illinois,(1963),195-209.

|8| ~~———————.~~ Irwin,J.M,and Peercy,C.L, and Walker,E.A. : High extensions of abelian groups.Acta Math.Acad.Sci Hungar,14,(1963)319-330.

|9| Hauptfleisch,G.J. : Splitting cristeria in the extension theory of groups. Tydskr Natuurwet,7,(1967), 358-365.

|10| ———————. : A note on cyclic extension, Nieuw Arch.Wisk(3) XV,(1967), 119-123.

|11| Honda, K. : Realism in the theory of abelian groups1.Comm. Math.Univ.St.Pauli, 5(1956),37-73.

|12| Irwin,J.M. : High subgroups ofabelian torsion groups. Pac.J.of Math.11.(1961), 1375-1384.

|13| ———————. Walker,E.A. : On N-high subgroups of abelian groups. Pac.J.Of Math.11,(1961), 1363-1374.

|14| ———————. : On isotype subgroups of abelian groups.Bull.Soc.Math.France,89, (1961),451-460.

|15| ———————. Peercy,C. and Walker,E.A. : Splitting properties of high sub-groups.Bull.Soc.Math.France,90, (1962),1-10.

|16| Kaplansky,I. : Infinite abelian groups,Uni.of Michigan Press, Ann.Arbor, Michigon,1954.

|17| Kuresh, A.G. : The theory of groups vol.1,2, english edition. Chelsea Publishing company, New York, (1960).

|18| Mashhood, A. : Note on neat-high extensions. Tamkang J. of Math. vol.13, No.2. (1982). (to appear).

|19| ——————. : On groups of neat and pure-high extensions. Tamkang J. of Math. vol.11, (1980), 83-90.

|20| ——————. : On pure-high subgroups of abelian groups. Kyungpook Math. J. vol. 22 No. 2 , (1982), (to appear).

|21| ——————. : On groups of neat extensions. Tamkang, J. of Math. vol.14, No.1, (1983). (to appear).

|22| Rangaswamy, K.M. : Extension theory of abelian groups. Math. Student; 32(1964), 11-16.

|23| ——————. : Characterisation of intersections of neat subgroups of abelian groups. J. Indian Math. Soc. 29, (1965), 31-36.

|24| Rotman, J.J. : The theory of groups: An introduction, Allyn And Bacon, Inc. Boston, Mass (1965).

|25| Simauti, K. : On abelian groups in which every neat subgroup is a pure subgroup. Comment. Math. Univ. St. Paul, 17, (1969), 105-110.

|26| Simauti,K. : On N-high subgroups of abelian p-groups. Comment. Math.Univ.St. Paul,16,(1967-68),1-3.

|27| Walker,C.P. : Relative homological algebra and abelian groups. Illinois J.Math. 10, (1966),186-209.

|28| Walker,E.A. : Torsion endomorphic images of mixed abelian groups. Pac.J.of Maths.vol.11,No.1,(1961),375-377.

|29| Yahya,S.M. : P-pure exact sequences and the group of p-pure extensions. Ann. Univ.Sci.Budapest5,(1962),179-191.

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